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A QUEUEING MODEL WITH FEEDBACK APPLIED TO A COMMUNICATION PROBLEM WITH INTERFERENCE-CAUSED RETRANSMISSIONS

Jörg Wilhelm Lüneburg

NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

A Queueing Model with Feedback Applied to a Communication Problem with Interference-Caused Retransmissions

by

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Thesis Advisor: Paul R. Milch

September 1972

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A Queueing Model with Feedback Applied to a Communication Problem with Interference-Caused Retransmissions

by

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Submitted in partial fulfillment of the
requirements for the degree of

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ABSTRACT

A queueing model with feedback is proposed as a mathematical representation of a communication system with interference-caused retransmissions. A satellite receives messages of constant length from ground stations. New messages arrive in a Poisson pattern with constant rate. A finite number of channels is available for transmission and channel selection is random among all channels. All messages with overlapping transmission times on the same channel have to be retransmitted. The arrival rate of messages is a composition of the constant arrival rate of the basic Poisson process and the arrival rate of the feedback from retransmissions. The arrival rate is approximated in discrete time steps. A necessary and sufficient condition for a finite limiting point of the resulting sequence of arrival rates is derived. Numerical results for finite limiting points, rate of convergence, and expected number of transmissions have been computed. Examples for sequences of arrival rates have been tabulated.

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I. INTRODUCTION

In the voluminous literature of Queueing Theory relatively little attention has been paid to queueing models with feedback. In these models customers have positive probability to apply for service repeatedly. Finch [Ref. 5] has examined cyclic queues with finitely many servers, finite waiting rooms, and constant probability of return to a specific server; so too has Takács [Ref. 9] for single server queues. Kopocińska [Refs. 7 and 8] has further researched the model by letting the probability of feedback be a function of the number of customers in the system. Dieter and Ahrens [Ref. 3] looked at Erlang's classic telephone problem, but did not consider calls arriving during busy periods as being lost or forming a queue. Instead, they assigned a positive probability to the possibility of repeated calls.

In the following a queueing model with feedback is proposed as a mathematical representation of a model for message transmission and interference-caused retransmission through a multi-channel satellite communication system.

In this model the probability of feedback will be a function of the arrival rate, which itself will depend on the constant arrival rate of new messages and the number of messages requiring retransmission. All messages are to be retransmitted until they have been received without interference.

II. THE PHYSICAL MODEL

A physical model of a satellite communication system is described in detail in Part IV (UNCLASSIFIED) of Criggler's thesis [Ref. 1] in which the probability of non-interference among messages transmitted over several channels is computed. The background for this system is classified US-SECRET and has not been available for this study; however, the details in the unclassified section sufficed to understand the stochastic problems of the model.

The model is briefly described as follows: A communication satellite receives messages from different ground stations, and relays them to a communication center. One or more channels are available for the traffic between the ground stations and the satellite. Channels are selected at random. The stations cannot detect whether a specific channel is busy at the moment, i.e., whether any other station is transmitting on that channel. The messages are coded and have constant length. A message will be received correctly and acknowledged by the receiver only if during its entire transmission time no other station is transmitting on the same channel. If two or more messages have overlapping transmission times on the same channel, each of the messages has to be retransmitted. The time from the beginning of a transmission to the time retransmission is commenced is constant.

The following assumptions are made:

(i) the number of ground stations is constant and finite;

(ii) every station will send one and only one original message;

(iii) all stations will begin transmission of their original message within a fixed time interval;

(iv) the starting points of the original messages are independent and uniformly distributed over this time interval;

(v) the messages are of constant length;

(vi) for any particular message the interval between the beginning of one transmission and the time retransmission is commenced is of constant length;

(vii) a constant number of channels is available;

(viii) channel selection is random among all channels for transmissions and retransmissions;

(ix) the communication net is a free net; i.e., no permission for transmitting is required.

Criggler has succeeded in arriving at an analytical solution for the probability of how many of the messages will be received correctly on their first transmission. He restricted his results in that he did not allow for retransmissions. In a further thesis based on Criggler's results Jones [Ref. 6] examined how the system develops if retransmissions are allowed. Jones made the assumption that only messages with the same number of transmissions and retransmissions can possibly interfere. This assumption restricts the applicability of the results obtained.

In order to avoid the above restriction a similar but different model has been examined which still offers a partial solution to the original problem.

Assumptions (i), (ii), (iii), and (iv) have been replaced by the following assumption: messages from ground stations arrive at a communication satellite according to a Poisson process with constant arrival rate. Assumptions (v) through (ix) remain unchanged.

III. PROBLEM

The physical model discussed above can be mathematically formulated as a queueing model. For this reason the terminology of queues will be used. The original messages are new customers, retransmissions are the feedback of customers, the satellite is the service facility, the channels are the servers, and the constant transmission times are the service times.

The queue discipline is as follows:

(i) customers arriving at the service facility select a server at random among all servers, they cannot distinguish between idle and busy servers;

(ii) if a customer selects a busy server, both the customer just being served and the customer arriving will proceed to a common waiting room with infinite capacity after their respective constant service times;

(iii) having waited for a minimum amount of time, (h), customers in the common waiting room will again choose a server at random and try to get undisturbed service;

(iv) a customer will leave the service facility only after he has received undisturbed service.

Thus, the total arrival rate of customers, f, is a composition of the constant arrival rate, f_o, of the basic Poisson process of arriving new customers and the feedback from the common waiting room. (See Figure 1).

This study has mainly been directed towards the behavior of the arrival rate, f, of customers requesting service. An approximate mathematical model is developed. A necessary and sufficient condition for feasibility of the system is derived.

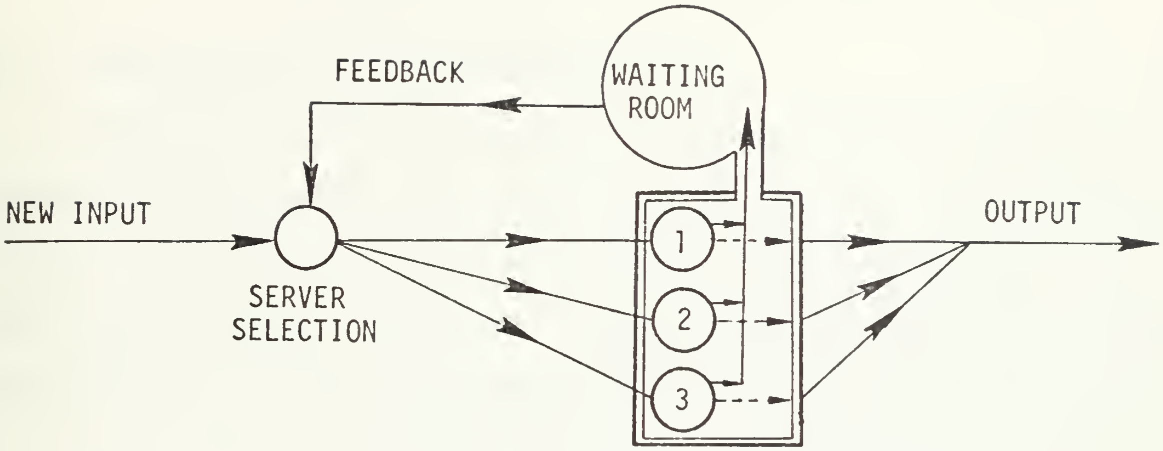

Figure 1. Service Facility with Three Servers and Feedback.

IV. DEVELOPMENT OF MATHEMATICAL MODEL

A. GENERAL DERIVATION OF MATHEMATICAL MODEL

Suppose new customers arrive at a service facility in a Poisson pattern with constant arrival rate, f_0. According to the model described above a certain number of customers will interfere with each other and will have to reenter for service. It is assumed that there is a minimum time interval, h, between any subsequent reentries of the same customer.

Thus, the arrival rate, f, will remain constant and equal to f_0 during the first interval (0,h]. Thereafter the arrival rate will increase.

As a discrete approximation it is assumed that the arrival rate for the interval (h, 2h] increases by $q(f_0)\ f_0$. Thus, the total arrival rate for this interval is $f_1 = f_0 + q(f_0)\ f_0$. $q(f_0)$ is the expected ratio of customers entering the waiting room while the arrival rate of customers is still f_0. This approximation is conservative because the increased arrival rate is assumed for the entire interval (h, 2h], although customers entering the waiting room at the end of the first interval (0, h] can apply for repeated service only after a minimum waiting time, h; i.e., at the end of the second interval.

In general, the arrival rate of the interval (j h, {j+1} h] will be: $f_j = f_0 + q(f_{j-1})\ f_{j-1}$.

The sequence of arrival rates, f_j, j = 0,1,..., can be computed successively from the equations:

$$
\begin{aligned}
f_0 &= f_0 \\
f_1 &= f_0 + q(f_0)\, f_0 \\
f_2 &= f_0 + q(f_1)\, f_1 \\
&\vdots \\
f_j &= f_0 + q(f_{j-1})\, f_{j-1} \qquad (1) \\
&\vdots
\end{aligned}
$$

This is equivalent to:

$$f_j = f_0 + q(f_{j-1})\, f_0 + q(f_{j-1})\, q(f_{j-2})\, f_0 + \cdots$$

$$+ q(f_{j-1}) \ldots q(f_0)\, f_0$$

$$= f_0 \left\{ 1 + \sum_{k=0}^{j-1} \left(\prod_{i=k}^{j-1} q(f_i) \right) \right\}$$

It is of major interest for the queueing model to consider under what condition the sequence $\{f_i\}$ converges to a finite limiting point, f^*.

B. EXPECTED RATIO OF CUSTOMERS REQUIRING REPEATED SERVICE

1. Service Facility with One Server

From the Poisson arrival pattern assumed in Section II it follows that the interarrival times between new customers are independent identically distributed random variables, X_i, with common c.d.f.

$$F_X(x) = P\,[X \le x] = \begin{cases} 1 - e^{-f_0 x} & , x \ge 0 \\ 0 & , x \le 0 \end{cases} \qquad (2)$$

Thus for any particular customer, T_i, the probability of receiving undisturbed service is equal to the probability that:

(i) he does not arrive within the service time of any previous customer;

(ii) no other customer arrives during his service time.

Let U_i be the event $\{T_i$ receives undisturbed service$\}$ and $\overline{U}_i$ be the event $\{T_i$ does not receive undisturbed service$\}$.

There is a positive probability that T_i interferes with any of the other customers T_j, $j \neq i$. If T_i interferes with T_j, $j > i$, then T_i also interferes with $T_{i+1}, T_{i+2}, \ldots, T_{j-1}$. Likewise, if T_i interferes with T_j , $j < i$, then T_i also interferes with $T_{i-1}, T_{i-2}, \ldots, T_{j+1}$. Therefore, the event $\{T_i$ interferes with T_j, $j \neq i\}$ is a subevent of the event $\{T_i$ interferes with T_{i-1} or $T_{i+1}\}$. Further, if T_i interferes with neither T_{i-1} nor T_{i+1}, then T_i does not interfere with any T_j, $j \neq i$. Thus, it is sufficient to know whether T_i interferes with T_{i+1} or T_{i-1}. The probability of event U_i is equal to the probability of the event $\{T_i$ interferes neither with T_{i+1} nor with $T_{i-1}\}$. Likewise the probability of the event $\overline{U}$ is equal to the probability of the event $\{T_i$ interferes with T_{i+1} or $T_{i-1}\}$. So:

$$P\,[U_i] = P[X_i > d \text{ and } X_{i+1} > d] = (P[X_i > d])^2$$

since X_i and X_{i+1} are independent and identically distributed random variables.

However, if it is known that T_{i-1} has received undisturbed service, it follows that:

$$P[U_i \mid U_{i-1}] = P[X_i > d \text{ and } X_{i+1} > d \mid X_{i-1} > d \text{ and } X_i > d]$$

$$= P[X_{i+1} > d]. \tag{3}$$

If it is known that customer T_{i-1} did not receive undisturbed service, it follows that:

$$P[U_i \mid \overline{U}_{i-1}] = P[X_i > d \text{ and } X_{i+1} > d \mid \overline{U}_{i-1}]$$

$$= \frac{P[(X_i > d \text{ and } X_{i+1} > d) \text{ and } (X_{i-1} \leq d \text{ and } X_i > d)]}{P[\overline{U}_{i-1}]}$$

$$= \frac{P[X_i > d]\, P[X_{i+1} > d]\, (1-P[X_{i-1} > d])}{1 - (P[X_{i-1} > d])^2}$$

$$= \frac{(P[X_i > d])^2}{1 + P[X_i > d]} \quad . \tag{4}$$

Equations (3) and (4) make it clear that the events U_i and U_{i-1} are dependent, and this dependence makes exact analytical results much too complicated. Instead, the expected ratio of customers receiving undisturbed service has been used to solve equation (1). Consider a fixed number of (n+1) successive interarrival times which define the arrival of n customers. (See Figure 2). Note that nothing has been assumed about the total time interval during which these customers arrive, so the interarrival times are still independent random variables with identical negative exponential distribution function. Let N_R be the random number of customers receiving undisturbed service. Let R be the quotient of N_R and the constant n. Since R is a function of a random variable it is itself a random variable,

$$R = \frac{N_R}{n} \quad .$$

Let the expected value of R be r,

$$r = E[R] = E\left[\frac{N_R}{n}\right] = \frac{1}{n}\; E[N_R] \quad . \tag{5}$$

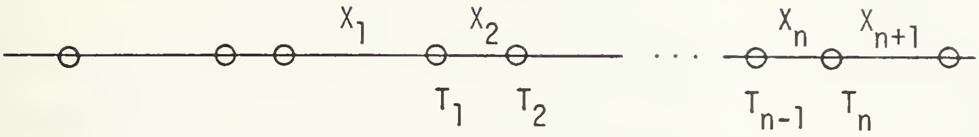

Figure 2. n Successive Customers Arriving at the Service Facility with Constant Arrival Rate.

The total number J of interarrival times greater than the constant service time, d, follows the binomial distribution with parameters (n+1, P[X > d]). Thus,

$$P[j \text{ out of } n+1 \text{ interarrival times are greater then } d]$$

$$= P_J[j]$$

$$= \binom{n+1}{j} \{P[X_i > d]\}^j \quad \{1-P[X_i > d]\}^{n+1-j}, \; j = 0,1,\ldots,n+1. \qquad (6)$$

These J events $\{X_i > d\}$ are partitioned into T runs, where T can take all values from zero to min (J, n+2-J). For every run of events $\{X_i > d\}$ of length K (i.e., exactly K consecutive interarrival times greater than d), there will be (K-1) customers receiving undisturbed service. So (J-T) customers will receive undisturbed service, given J events $\{X_i > d\}$ are partitioned into T runs. This implies that:

$$P[t \text{ runs of events } \{X_i > d\} \mid n,J]$$

$$= \; P[J\text{-}t \text{ customers receive undisturbed service} \mid n,J]. \qquad (7)$$

This probability is shown by Feller [Ref. 4] to be:

$$P_T[t|n,J] = \binom{J-1}{t-1}\binom{n+2-J}{t} \Big/ \binom{n+1}{J} , \quad t=0,1,\ldots,\min(J,n+2-J). \quad (8)$$

The expected number out of n customers receiving undisturbed service is therefore:

$$E[N_R] = E_J[E_T[E[N_R|J,T]|J]]$$

$$= E_J[E_T[J-T|J]] \qquad \text{(from 7)}$$

$$= E_J\left[\frac{J(J-1)}{n+1}\right] \qquad \text{(see Appendix A)}$$

$$= n(P[X_i > d])^2 \qquad \text{(see Appendix B)} \quad .$$

The expected ratio of customers receiving undisturbed service is:

$$r = \frac{1}{n} E[N_R]$$

$$= (P[X_i > d])^2$$

and the ratio of customers requiring repeated service is:

$$q(f_o) = 1 - r$$

$$= 1 - (P[X_i > d])^2 \ .$$

For this model, $P[X_i > d]$ is given by (2) and therefore:

$$q(f_o) = 1 - e^{-2df_o} \quad . \qquad (9)$$

2. Service Facility with c Servers

Extending the model to c servers the additional assumption is made that each arriving customer selects a server at random, independent of other customers. So P[arriving customer selects server C_i] = $\frac{1}{c}$, $i = 1,2,\ldots,c$. The model assumes that only customers served by the same

server can interfere. Looking at one particular server, C_i, the interarrival times of customers are sums of i.i.d. random variables. If $(Z_i - 1)$ is the number of customers arriving between two consecutive customers choosing server C_i, then Z_i is a random variable with geometric $(\frac{1}{c})$ distribution. The range of Z_i is 1, 2, ... Therefore, the interarrival time Y_{ij} between two consecutive customers who choose server C_i is the sum of Z_i ordinary arrival times,

$$Y_{ij} = \sum_{k=1}^{Z_i} X_k \quad .$$

From (2) it follows that Y_{ij} is distributed Erlang (f_o) of order z_i, given $Z_i = z_i$.

The moment generating function (m.g.f.) of Y_{ij} is:

$$\begin{aligned}\psi_{Y_{ij}}(v) &= E[e^{vY_{ij}}] \\ &= E[E[e^{vY_{ij}}|Z_i]] \\ &= E\left[\left(\frac{f_o}{f_o-v}\right)^{Z_i}\right]\end{aligned}$$

Let $\frac{f_o}{f_o-v} = e^s$, then:

$$\begin{aligned}\psi_{Y_{ij}}(v) &= E[e^{sZ_i}] \\ &= \frac{pe^s}{1-qe^s} \quad , \text{ where } \quad p = \frac{1}{c}, \ q = 1 - p \\ &= \frac{p\left(\frac{f_o}{f_o-v}\right)}{1 - q\left(\frac{f_o}{f_o-v}\right)} \\ &= \frac{pf_o}{pf_o-v} \qquad (10)\end{aligned}$$

Equation (10) is the m.g.f. of the exponentially distributed random variable with parameter pf_o, therefore

$$Y_{ij} \text{ is distributed exponentially with parameter } \frac{1}{c} f_o. \tag{11}$$

The interarrival times at any particular server are again exponentially distributed, and the results obtained in B.1. still hold with the only change that the parameter f_o is replaced by $\frac{1}{c} f_o$.

The expected ratio of customers receiving undisturbed service from server C_i, given N_i customers have arrived at this server, is therefore:

$$\begin{aligned} r_i &= \frac{1}{N_i} E[N_{R_i} \mid N_i] \\ &= e^{-\frac{2df_o}{c}} \qquad \text{(from Equations (9) and (11))} \end{aligned} \tag{12}$$

To extend the result to the total system of c servers, without regard to how many customers have been served by any particular server, the expected ratio r for the total system will next be derived.

The n arriving customers are partitioned by the system of c servers into c groups. The resulting random vector $(N_1, N_2, \ldots, N_c)$ has a multinomial probability distribution. From Equation (5) it is known that:

$$\begin{aligned} r &= \frac{1}{n} E[N_R] \\ &= \frac{1}{n} E[N_{R_1} + N_{R_2} + \ldots + N_{R_c}] \\ &= \frac{1}{n} \sum_{i=1}^{c} E[N_{R_i}] \end{aligned}$$

$$= \frac{1}{n} \sum_{i=1}^{c} E[E[N_{R_i} | N_i]]$$

$$= \frac{1}{n} \sum_{i=1}^{c} E[N_i \; e^{-\frac{2df_o}{c}}] \quad \text{from Equation (12)}$$

$$= \frac{1}{n} \; e^{-\frac{2df_o}{c}} \sum_{i=1}^{c} E[N_i]$$

$$= e^{-\frac{2df_o}{c}} , \tag{13}$$

because $\sum_{i=1}^{c} E[N_i] = E[\sum_{i=1}^{c} N_i] = n$.

The expected ratio of customers requiring repeated service is:

$$q(f_o) = 1 - e^{-\frac{2df_o}{c}} \tag{14}$$

This result shows that it will be sufficient to study the single server queueing model. Any results can easily be extended to the multiple server queueing model by redefining f_o as the arrival rate of new customers at any particular server, i.e.,

$$f_o' = f_o \qquad \text{for single server model,}$$

$$f_o' = \frac{f_o}{c} \qquad \text{for model with c servers} \quad .$$

The constant service time, d, allows further simplification. The arrival rate f' can be chosen to be the expected number of customers arriving during an interval of length 2d instead of the unit interval.

Equation (14) then becomes:

$$q(f_0) = 1 - e^{-f_0} , \tag{15}$$

where f_0 is written in place of f'_0 .

C. THE SEQUENCE OF ARRIVAL RATES

The sequence $\{f_i\}$ is strictly monotone increasing. (See Appendix C.) Two possible cases can be distinguished.

Case 1: the sequence converges to a finite limiting point, f*.

Case 2: the sequence converges to infinity.

If the sequence converges to a finite limiting point, f*, the problem will reduce to a queueing problem with constant arrival rate, f*, after convergence. The state of the queueing model will be defined as transient before convergence and as steady thereafter.

If the sequence converges to infinity the system will eventually become infeasible, because the probability of any customer to receive undisturbed service will go to zero.

1. Necessary and Sufficient Condition for Convergence to a Finite Limiting Point

The sequence $\{f_i\}$ converges to a finite limiting point f* if and only if:

$$f_0 \le e^{-1} \tag{16}$$

is satisfied.

Proof:

(i) Necessary condition.

If the sequence $\{f_i\}$ converges to a finite limiting point, f*, it must be true from equations (1) and (15) that:

$f^* = f_0 + f^* (1 - e^{-f^*})$, or equivalently

$$f_0 = f^* e^{-f^*} . \tag{17}$$

Define the function $H(f) = f\,e^{-f}$, $f > 0$. (See Figure 3). The absolute maximum of $H(f)$ is the upper bound for f_0 for which equation (17) can still be satisfied. $H(f)$ is strictly positive throughout its range except at the boundary points of its variable f, zero and infinity, where it is equal to zero, i.e., $H(0) = H(\infty) = 0$. Since $H(f)$ is a continuous function it will have at least one relative maximum value for some $f \,\varepsilon(0,\infty)$. From the sufficient conditions for a relative maximum, i.e., $H'(f_{max}) = 0$ and $H''(f_{max}) < 0$, $f_{max} = 1$ and $H(f_{max}) = e^{-1}$. Since this is the only relative maximum for $f \,\varepsilon(0,\infty)$ this is also the absolute maximum value of $H(f)$.

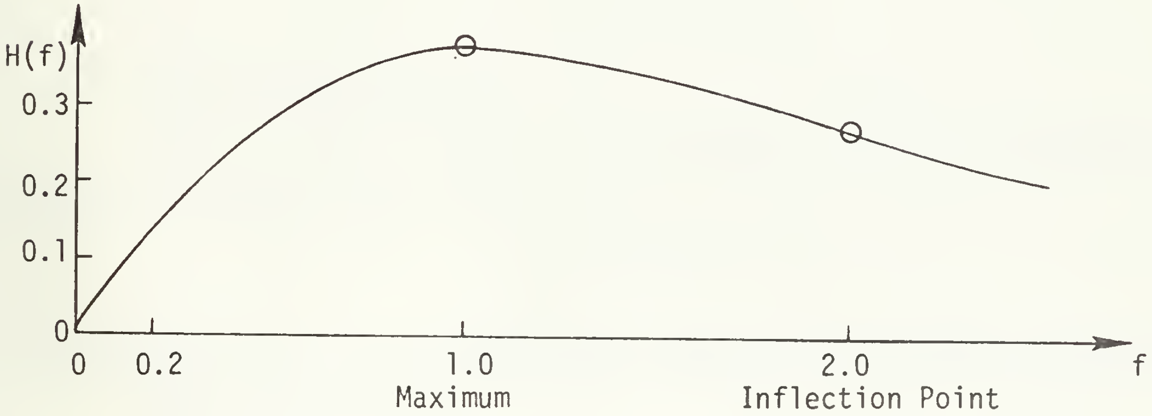

Figure 3. The Function $H(f) = f\,e^{-f}$, $f > 0$.

Thus, the necessary condition for a finite limiting point f^* of the sequence $\{f_i\}$ is:

$$f_0 \leq e^{-1} \quad ,$$

since only then will equation (17) have a solution for f^*.

(ii) Sufficient condition.

Given condition (16) it will be shown by induction that f* is bounded from above by one and so the sequence converges to a finite limiting point:

$$f_o < 1 \qquad \text{(from Equation (16))}$$

Assuming that $f_n < 1$ it has to be proven that $f_{n+1} < 1$:

$$\begin{aligned} f_{n+1} &= f_o + f_n(1 - e^{-f_n}) && \text{(from Equation (1))} \\ &< f_o + 1(1 - e^{-1}) && \text{(from the above assumption)} \\ &\leq e^{-1} + (1 - e^{-1}) && \text{(from Equation (16))} \\ &= 1 \ . \end{aligned}$$

End of Proof.

2. The Limiting Point f*

Given that the necessary and sufficient condition is satisfied, f* can be evaluated from Equation (17):

$$f_o = f^* e^{-f^*} \ .$$

This equation can be solved by Newton's method of successive approximation.

3. Rate of Convergence

Given the value of f_o and having calculated the limiting point f* the number M can be derived such that, given some $\varepsilon > 0$,

$$(f^* - f_m) < \varepsilon, \text{ for all } m > M.$$

M can be found in two ways:

(i) by repeated calculation of f_i, $i=1,2,\ldots,M$, where the calculation is terminated when $(f^* - f_M) \leq \varepsilon$;

(ii) by utilizing the difference function G(f), where G(f) is defined as $G(f_i) = f_{i+1} - f_i$

$$= f_o - f_i \, e^{-f_i} \qquad \text{(From Equation 1).}$$

$G(f)$ has its absolute minimum at $f_{min} = 1$. If $\{f_i\}$ converges to f^* this minimum will be less than or equal to zero, and $\min_f\{G(f) = 0\} = f^*$. Starting at f_0 and adding the value of $G(f_i)$ to the current value of f_i will result in f_{i+1}. The number of steps necessary to get within ε - distance of f^* is M.

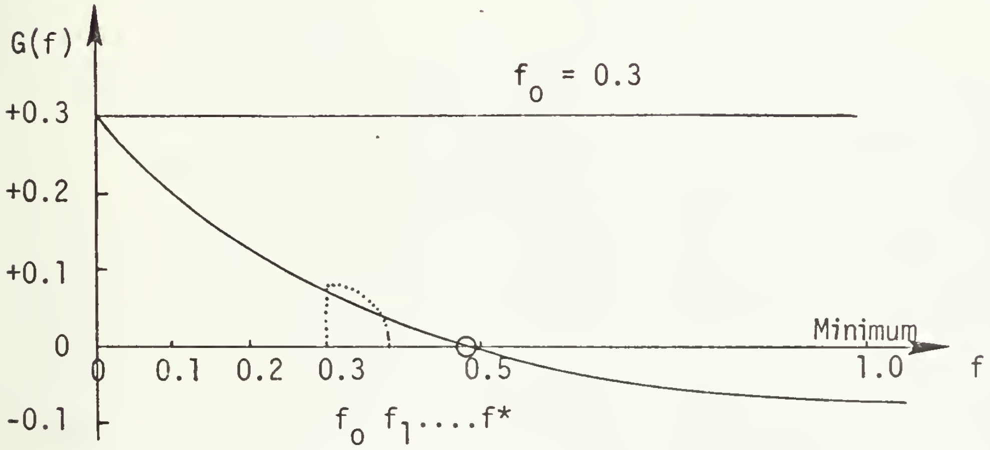

Figure 4. The Difference Function G(f).

4. Expected Number of Tries for any Particular Customer

If the sequence $\{f_i\}$ converges to a finite limiting point the expected number of tries for a particular customer arriving during the steady state of the system can be derived. Let the random variable W be the number of tries of a particular customer before he receives undisturbed service.

Define p_w = P[customer T_i gets undisturbed service on his w^{th} try].

From Section IV.B. it follows in steady state that:

$$
\begin{aligned}
p_0 &= 0 \\
p_1 &= e^{-f^*} \\
p_2 &= (1 - e^{-f^*})\, e^{-f^*} \\
&\vdots \\
p_w &= (1 - e^{-f^*})^{w-1}\, e^{-f^*} \\
&\vdots
\end{aligned}
$$

Thus W is a geometrically distributed random variable with parameter p_1. Its expected value is:

$$E[W] = e^{f^*} \quad . \tag{19}$$

5. The Case of Infinite Limiting Point

Given that the necessary and sufficient condition for convergence to a finite limiting point is not satisfied, i.e., $f_0 > e^{-1}$, there will be no solution to the difference function such that $G(f_i) = 0$. This means that the difference between f_{i+1} and f_i, $i = 0,1,\ldots$, will always be greater than or equal to some $\delta > 0$, where $\delta = G(f_{min}) = G(1) > 0$.

To keep the system feasible, i.e., to keep the arrival rate finite, admission of new customers has to be interrupted from time to time until the customers in the waiting room have been served.

The following stopping event was chosen in numerical computations: {admission of new customers will be stopped if $G(f_i) > G(f_1)$}. This means that the difference between two successive elements of the sequence $\{f_i\}$, namely elements f_i and f_{i-1}, has increased beyond the difference of the first two elements, f_1 and f_0. No new customers will be allowed after the stopping event, the sequence will therefore become $\{f_i\}$ such that:

$$f_{i+1} = f_i\, q(f_i) \quad .$$

This sequence is monotone decreasing and converges to zero, since:

$$q(f_i) = 1 - e^{-f_i}, \quad f_i > 0$$

is positive and strictly less than one.

Two cases can be distinguished:

(i) Case 1: $e^{-1} < f_o \le 1$. The stopping event will occur after some finite number of elements of the sequence $\{f_i\}$. (See Figure 5.)

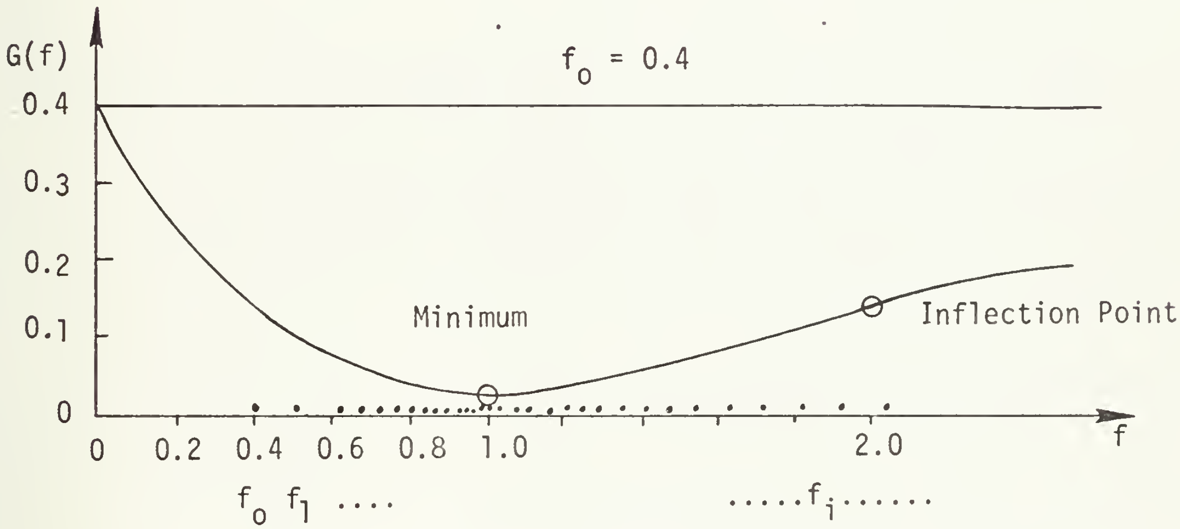

Figure 5. The Difference Function G(f), $e^{-1} < f_o \le 1$.

(ii) Case 2: $f_o > 1$. In this case $G(f_2)$ is already greater than $G(f_1)$, since f_o lies to the right of the minimum of the difference function G(f), $f_{min} = 1$. (See Figure 6.)

New customers can be admitted again after the arrival rate from the waiting room has dropped below some $\varepsilon > 0$.

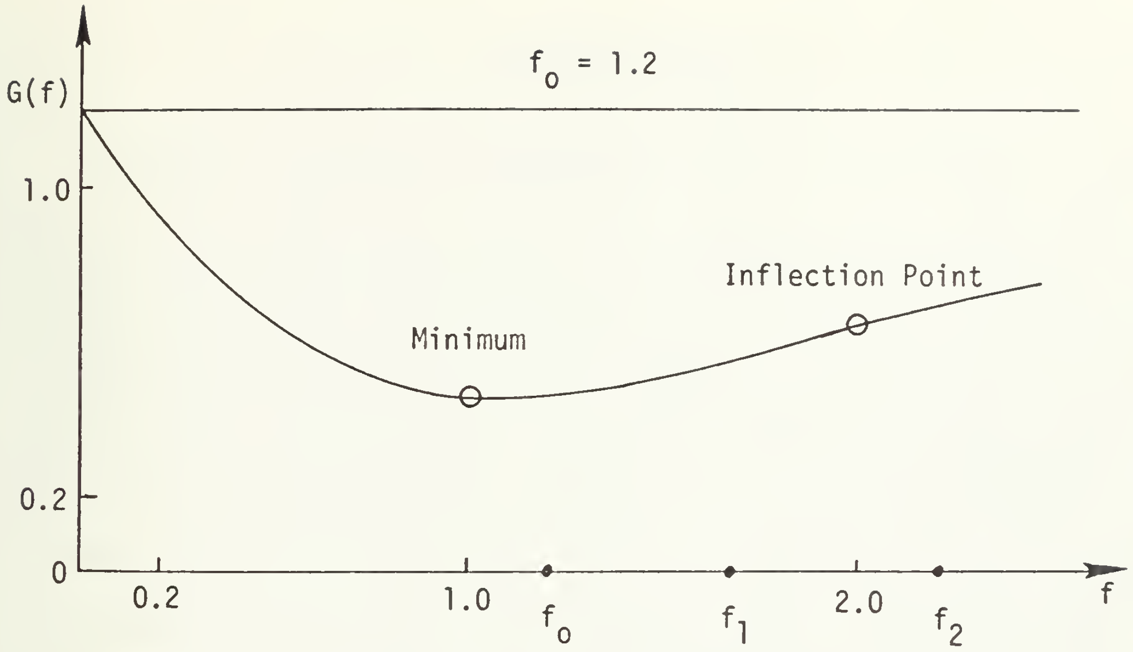

Figure 6. The Difference Function G(f), $f_o > 1$.

V. NUMERICAL RESULTS

Numerical results were computed for constant arrival rates, f_0. The values of f_0 ranged from 0.01 to 1.00 arrivals per time interval of length 2d in steps of 0.01. These results can easily be extended to constant arrival rates per unit time interval, λ_0, and constant service time, d.(E.g., $\lambda_0 = 0.02$ and $d = 2$ is equivalent to $f_0 = 2d\lambda_0 = 0.08$). For each of the arrival rates f_0 the sequence $\{f_i\}$ was calculated.

For $f_0 \leq e^{-1}$ the limiting point f* was computed by Newton's method of successive approximation. The expected number of repeated tries for a particular customer arriving during the steady state of the system was tabulated.

For $f_0 > e^{-1}$ the sequence $\{f_i\}$ was computed with the stopping event as described in Section IV.C.5. The intervals with/without admission of new customers were tabulated.

Computations were done on the IBM 360 computer of the Naval Postgraduate School, Monterey, California.

Numerical results are tabulated in Tables I through III. (See Computer Output).

A description of the variables computed and a comprehensive graphical representation of the numerical results obtained is given in the following Sections (i) through (iv).

(i) The limiting points, f*, were computed by Newton's method of successive approximation from equation (17):

$$f_o = f^* e^{-f^*} .$$

Computation was stopped when the difference between successive approximations became less than $\varepsilon = 10^{-7}$.

(See computer output Table III and Figure 7.)

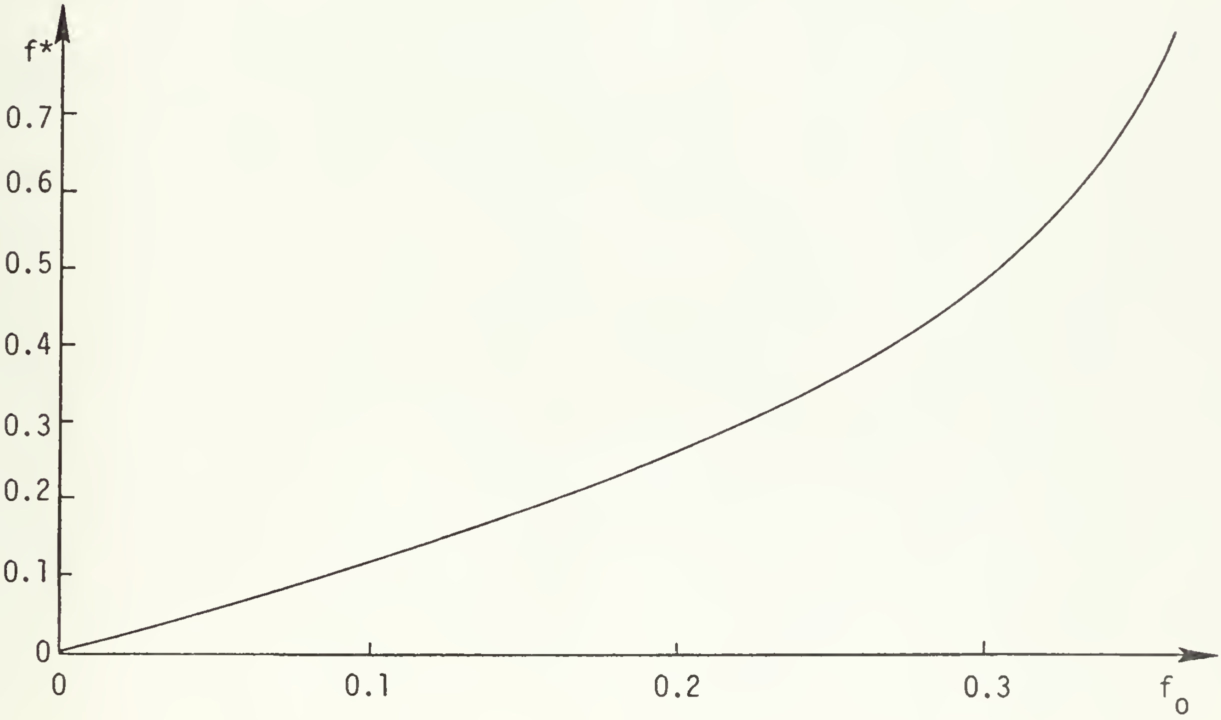

Figure 7. Limiting Points f*.

(ii) The expected number of repeated tries for a particular customer arriving during the steady state of the system was computed from equation (19):

$$E\,[W] = e^{f^*} .$$

The values used for f* were obtained from section V.(i). (See computer output Table III and Figure 8.)

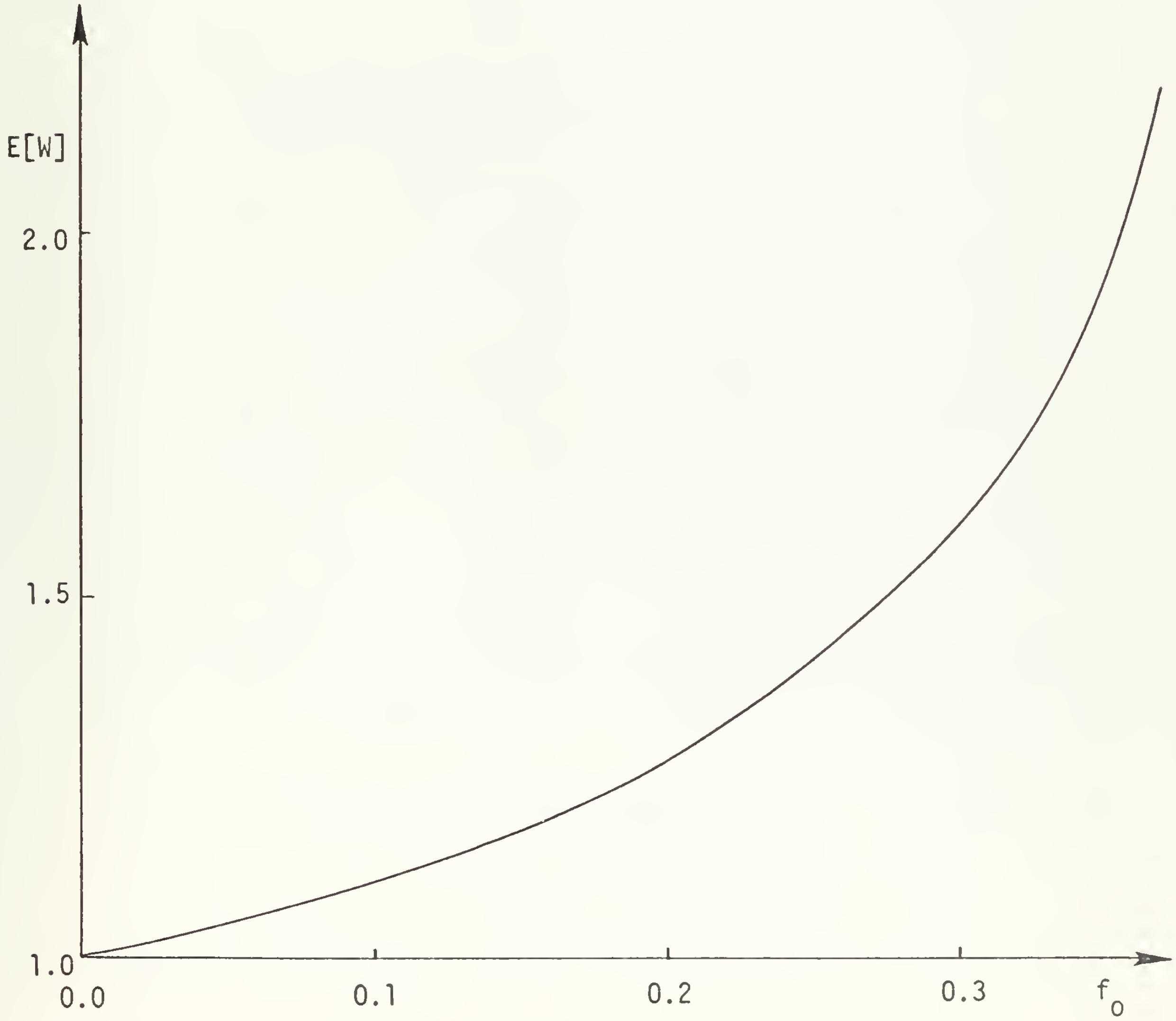

Figure 8. Expected Number of Transmission/Retransmissions During Steady State.

(iii) The rate of convergence was measured by the number M of elements of the sequence $\{f_i\}$ necessary to reach the limiting point f* within an $\varepsilon = 10^{-7}$. (See computer output Table I and Figure 9.)

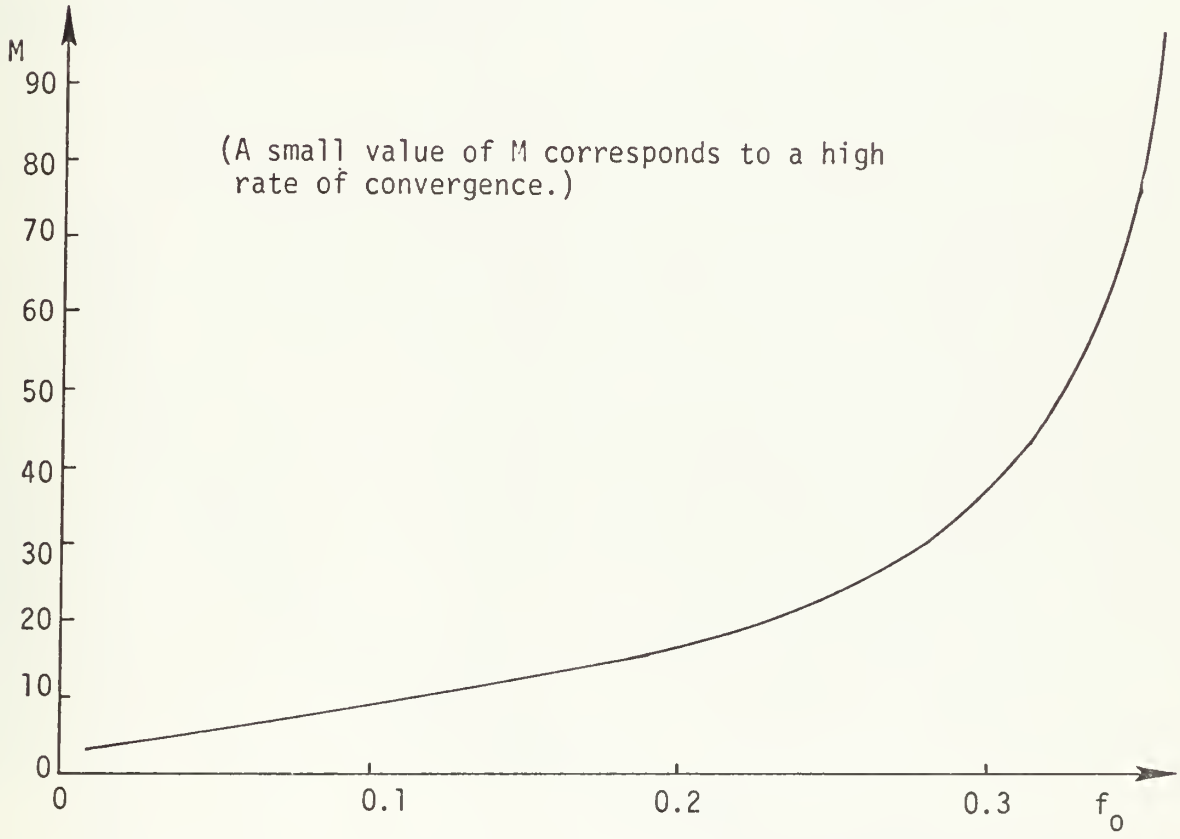

Figure 9. Rate of Convergence.

(iv) For systems with infinite limiting point two phases were distinguished:

- the active phase during which new customers were admitted, and
- the passive phase during which only customers arriving from the waiting room were admitted.

Stopping event for the active phase was: $f_i - f_{i-1} > f_1 - f_o$; stopping event for the passive phase was $f_i \leq \gamma = 10^{-7}$. (See computer output Table II and Figure 10.)

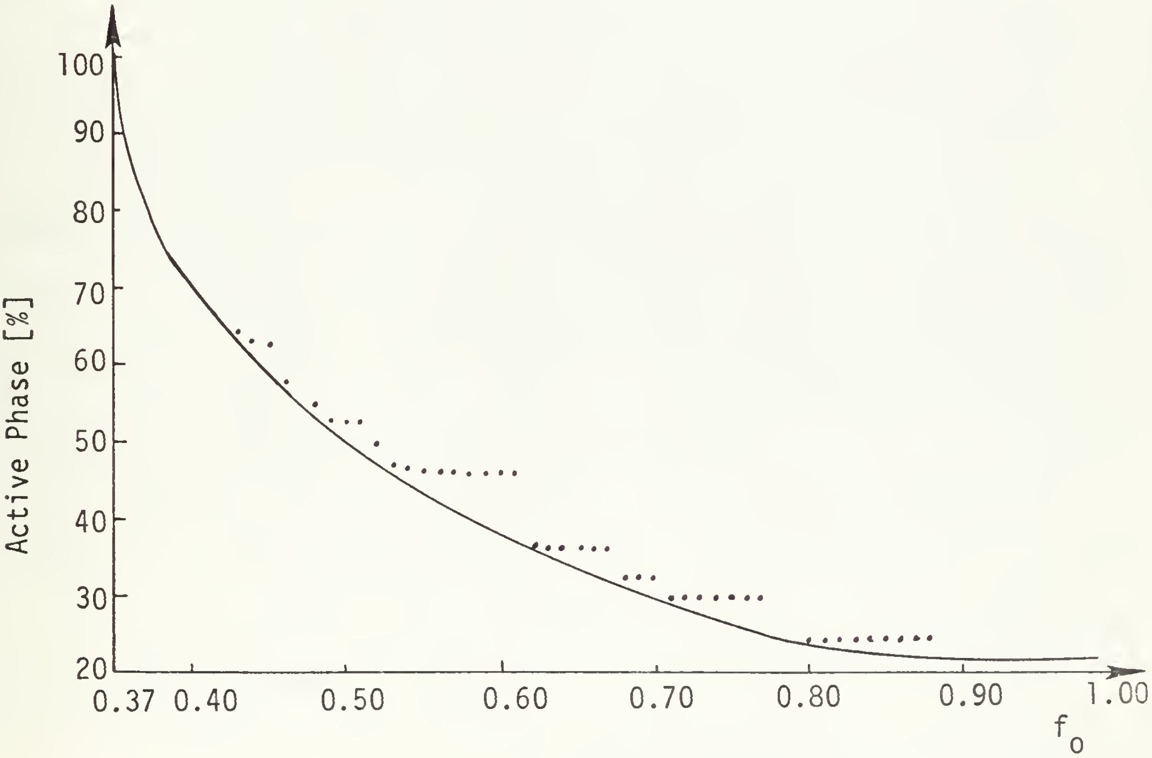

Figure 10. System with Interrupted Admission of New Customers-Percentage of Active Phase of the System, $e^{-1} < f_o \leq 1$.

VI. CONCLUSIONS AND RECOMMENDATIONS

For the queueing model with feedback considered in this paper a condition has been derived under which the arrival rate is expected to settle down to a limiting value. The input consists of new customers and the feedback of customers who did not receive undisturbed service.

If this condition is violated the input will increase ad infinitum and so will the queue length. To make the system feasible, interrupted admission has to keep new customers from entering the service facility to allow for the queue in the waiting room to be served first.

In terms of the physical problem which gave rise to this model, the following conclusions can be drawn and recommendations be made.

The communication system as it is designed can be expected to remain feasible as long as $f_0 \le e^{-1} = 0.368$, where f_0 is the arrival rate of new messages for a time interval of length 2d. In this case the incoming messages, including necessary retransmissions, will converge to a Poisson process with constant rate $f^* = 2d\lambda^*$. (λ = arrival rate per unit time interval.) The rate of convergence is proportional to the difference $(e^{-1} - f_0)$. (See Figure 9.)

The expected number of messages arriving during the steady state of the system is shown in Figure 8.

If the condition $f_0 \le e^{-1}$ is not satisfied, there are three possible ways to keep the system working:

(i) a terminating event has to be defined after which no new messages may be transmitted until all retransmissions are received undisturbed. In most practical cases, however, this will result in an

increase in the arrival rate immediately after the blockage is lifted, thus making the situation worse than it was. Due to strict communication policies in most military nets only vital messages will be transmitted. Prohibiting any new transmissions will, therefore, in general not make those messages obsolete which would have been transmitted otherwise. Therefore, the messages will form a queue at their ground stations and increase the original arrival rate, f_o, after transmission is allowed again.

(ii) Decreasing the message length may be possible to a certain degree. Extensive use of common formats and codes will improve the situation; however, it must be weighed against the additional time necessary to set up and read messages in this form. A decrease in transmission time may also be gained by use of automated transmission rather than voice or manual transmission. This can be achieved by recording the message at normal speed and transmitting it from the recorder at high speed. This method, used during World War II by German submarines on war patrol, has the additional advantage of reducing the detectability of the ground station. The disadvantage of this method is that it requires additional time for message preparation and handling at both ends.

(iii) Decreasing the arrival rate of new messages on a particular channel can be obtained by either enforcing restrictions on the type of messages allowed in the system, by reducing the number of ground stations in a particular net, or by increasing the number of transmission channels.

Based on the numerical results of the model, this system is recommended only for communication nets with light traffic. It follows

from the condition for convergence to a finite arrival rate that new messages can arrive at a maximum rate of 0.368 per time interval of length 2d, i.e., a maximum of 18.4 percent of the total time can be allocated for new messages on each channel.

If the traffic is expected to be heavier, a directed net may be instituted. Here stations have to request permission to transmit and are then given time and channel assignment by a net control station. In this net the load on every channel can be increased to approximately 80 percent allowing 20 percent for spacing between messages. The original model will still be of interest for this directed net since the service channel for requests and acknowledgements will always act as a free net. An advantage for this service channel will be the short transmission times necessary for requests and acknowledgements thus allowing for a higher arrival rate.

A final word of caution: the expected number of received messages is used in this queueing model to approximate the probability that a message will be received undisturbed. This approach is optimistic. A more conservative approach would be to increase the expected value by a factor of one or two standard deviations.

The general results are expected to be similar but more restrictive on the necessary and sufficient condition for convergence. The analytical part becomes fairly complicated.

A computer simulation could give further insight into this part of the problem.

APPENDIX A

$$E_T[J-T \mid J] = \frac{J(J-1)}{n+1}$$

Proof:

$$E_T[J-T \mid J] = \sum_{T=1}^{J} (J-T) \; P[T]$$

$$= \sum_{T=1}^{J} (J-T) \binom{J-1}{T-1}\binom{n+2-J}{T}\bigg/\binom{n+1}{J} \quad \text{(from Equation (8))}$$

$$= J \sum_{T=1}^{J} \binom{J-1}{T-1}\binom{n+2-J}{T}\bigg/\binom{n+1}{J} \qquad \text{(A)}$$

$$- \sum_{T=1}^{J} T \binom{J-1}{T-1}\binom{n+2-J}{T}\bigg/\binom{n+1}{J} \qquad \text{(B)}$$

(A) is the probability mass function of T, summed over all possible values[1] of T, multiplied by J, therefore (A) = J.

(B) is the expected value of T, shown in [Ref. 2] to be:

$$E[T] = \frac{J(n+2-J)}{n+1} .$$

$$E_T[J-T \mid J] = (A) - (B) = J - \frac{J(n+2-J)}{n+1} = \frac{J(J-1)}{n+1} .$$

End of Proof.

---

[1] If $\min(J, n+2-J) = n+2-J$, then $\binom{n+2-J}{T} \equiv 0$ for $T > n+2-J$.

APPENDIX B

$$E[N_R] = n(P[X_i > d])^2$$

Proof:

$$E[N_R] = E\left[\frac{J(J-1)}{n+1}\right] \qquad \text{(from Appendix A)}$$

$$= \frac{1}{n+1} \sum_{j=0}^{n+1} j(j-1) \binom{n+1}{j} (P[X_i > d])^j \, (1-P[X_i > d])^{n+1-j} \; .$$

(from Equation (6))

This is $\frac{1}{n+1}$ times the second factorial moment of the binomial distribution with parameters $(n+1,\ P[X_i > d])$, which is shown in [Ref. 2] to be:

$$(n+1)\ n\ (P[X_i > d])^2 \; .$$

Thus

$$E[N_R] = n\ (p[X_i > d])^2 \; .$$

End of Proof

APPENDIX C

The sequence $\{f_i\}$ is strictly monotone increasing, i.e., $f_{i+1} > f_i$, $i = 0,1,\ldots$, where $f_o > 0$ and $f_{i+1} = f_o + f_i\,(1 - e^{-f_i})$, $i = 0, 1, \ldots$.

Proof by induction:

$f_1 > f_o$, since $f_1 - f_o = (1 - e^{-f_o})\, f_o > 0$ for $f_o > 0$.

Assuming that $f_r > f_{r-1}$, it follows that:

$$\begin{aligned} f_{r+1} &= f_o + (1 - e^{-f_r})\, f_r \\ &> f_o + (1 - e^{-f_{r-1}})\, f_{r-1} \\ &= f_r \;, \end{aligned}$$

since $f(1 - e^{-f})$ is strictly monotone increasing for finite $f > 0$, and $f_r > f_{r-1}$ by assumption.

End of proof.

```
TABLE I     PAGE  1
SEQUENCE OF ARRIVAL RATES (FINITE LIMITING POINT)
         (ZEROS : NO ENTRY)

 F(  0)=   0.0100000 0.0200000 0.0300000 0.0400000 0.0500000
 F(  1)=   0.0100995 0.0203960 0.0308866 0.0415684 0.0524385
 F(  2)=   0.0101015 0.0204118 0.0309394 0.0416925 0.0526790
 F(  3)=   0.0101015 0.0204124 0.0309426 0.0417025 0.0527033
 F(  4)=   0.0000000 0.0204124 0.0309428 0.0417033 0.0527057
 F(  5)=   0.0000000 0.0000000 0.0309428 0.0417034 0.0527060
 F(  6)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0527060
```

```
TABLE I     PAGE  2
SEQUENCE OF ARRIVAL RATES (FINITE LIMITING POINT)
        (ZEROS : NO ENTRY)

 F( 0)=   0.0600000 0.0700000 0.0800000 0.0900000 0.1000000
 F( 1)=   0.0634941 0.0747324 0.0861506 0.0977461 0.1095162
 F( 2)=   0.0639061 0.0753813 0.0871112 0.0991021 0.1113603
 F( 3)=   0.0639562 0.0754734 0.0872672 0.0993502 0.1117355
 F( 4)=   0.0639623 0.0754865 0.0872927 0.0993959 0.1118125
 F( 5)=   0.0639631 0.0754884 0.0872968 0.0994044 0.1118284
 F( 6)=   0.0639632 0.0754887 0.0872976 0.0994059 0.1118317
 F( 7)=   0.0639632 0.0754887 0.0872977 0.0994062 0.1118323
 F( 8)=   0.0000000 0.0000000 0.0872977 0.0994063 0.1118324
 F( 9)=   0.0000000 0.0000000 0.0000000 0.0000000 0.1118325
```

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TABLE I    PAGE  3
SEQUENCE OF ARRIVAL RATES (FINITE LIMITING POINT)
        (ZEROS : NO ENTRY)


 F( 0)=   0.1100000 0.1199999 0.1299999 0.1399999 0.1499999
 F( 1)=   0.1214582 0.1335695 0.1458475 0.1582897 0.1708937
 F( 2)=   0.1238914 0.1367006 0.1497930 0.1631731 0.1768454
 F( 3)=   0.1244363 0.1374660 0.1508382 0.1645666 0.1786649
 F( 4)=   0.1245596 0.1376555 0.1511192 0.1649711 0.1792319
 F( 5)=   0.1245877 0.1377026 0.1511951 0.1650890 0.1794097
 F( 6)=   0.1245940 0.1377143 0.1512156 0.1651235 0.1794655
 F( 7)=   0.1245955 0.1377172 0.1512212 0.1651335 0.1794831
 F( 8)=   0.1245958 0.1377179 0.1512226 0.1651365 0.1794886
 F( 9)=   0.1245959 0.1377181 0.1512231 0.1651373 0.1794903
 F(10)=   0.0000000 0.1377181 0.1512232 0.1651375 0.1794909
 F(11)=   0.0000000 0.0000000 0.1512232 0.1651376 0.1794910
 F(12)=   0.0000000 0.0000000 0.0000000 0.0000000 0.1794911
```

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TABLE I     PAGE  4
SEQUENCE OF ARRIVAL RATES (FINITE LIMITING POINT)
         (ZEROS : NO ENTRY)

 F( 0)=   0.1599999 0.1699999 0.1799999 0.1899999 0.1999999
 F( 1)=   0.1836569 0.1965768 0.2096512 0.2228776 0.2362537
 F( 2)=   0.1908136 0.2050813 0.2196518 0.2345279 0.2497122
 F( 3)=   0.1931468 0.2080258 0.2233156 0.2390293 0.2551800
 F( 4)=   0.1939240 0.2090700 0.2246937 0.2408195 0.2574722
 F( 5)=   0.1941847 0.2094434 0.2252170 0.2415392 0.2584453
 F( 6)=   0.1942723 0.2095772 0.2254165 0.2418298 0.2588604
 F( 7)=   0.1943018 0.2096252 0.2254926 0.2419474 0.2590379
 F( 8)=   0.1943118 0.2096425 0.2255216 0.2419949 0.2591139
 F( 9)=   0.1943151 0.2096487 0.2255327 0.2420142 0.2591465
 F(10)=   0.1943163 0.2096509 0.2255369 0.2420220 0.2591604
 F(11)=   0.1943166 0.2096517 0.2255386 0.2420251 0.2591664
 F(12)=   0.1943167 0.2096520 0.2255392 0.2420264 0.2591689
 F(13)=   0.1943168 0.2096521 0.2255394 0.2420270 0.2591700
 F(14)=   0.0000000 0.2096522 0.2255395 0.2420272 0.2591705
 F(15)=   0.0000000 0.0000000 0.0000000 0.2420273 0.2591707
 F(16)=   0.0000000 0.0000000 0.0000000 0.2420273 0.2591708
 F(17)=   0.0000000 0.0000000 0.0000000 0.0000000 0.2591708
```

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TABLE I     PAGE   5
SEQUENCE OF ARRIVAL RATES (FINITE LIMITING POINT)
        (ZEROS : NO ENTRY)


 F( 0)=   0.2099999 0.2199998 0.2299998 0.2399998 0.2499998
 F( 1)=   0.2497771 0.2634456 0.2772570 0.2912090 0.3052995
 F( 2)=   0.2652070 0.2810140 0.2971349 0.3135710 0.3303233
 F( 3)=   0.2717806 0.2888433 0.3063802 0.3244029 0.3429226
 F( 4)=   0.2746778 0.2924621 0.3108515 0.3298727 0.3495526
 F( 5)=   0.2759728 0.2941613 0.3130523 0.3326890 0.3531166
 F( 6)=   0.2765552 0.2949650 0.3141446 0.3341531 0.3550537
 F( 7)=   0.2768179 0.2953464 0.3146890 0.3349180 0.3561126
 F( 8)=   0.2769365 0.2955276 0.3149608 0.3353186 0.3566933
 F( 9)=   0.2769901 0.2956139 0.3150967 0.3355287 0.3570124
 F(10)=   0.2770143 0.2956549 0.3151647 0.3356389 0.3571879
 F(11)=   0.2770252 0.2956744 0.3151987 0.3356968 0.3572844
 F(12)=   0.2770301 0.2956837 0.3152157 0.3357272 0.3573375
 F(13)=   0.2770323 0.2956882 0.3152242 0.3357431 0.3573667
 F(14)=   0.2770333 0.2956902 0.3152285 0.3357515 0.3573828
 F(15)=   0.2770338 0.2956913 0.3152306 0.3357559 0.3573917
 F(16)=   0.2770340 0.2956917 0.3152317 0.3357582 0.3573965
 F(17)=   0.2770341 0.2956920 0.3152322 0.3357594 0.3573992
 F(18)=   0.2770342 0.2956921 0.3152325 0.3357601 0.3574007
 F(19)=   0.0000000 0.2956921 0.3152326 0.3357604 0.3574015
 F(20)=   0.0000000 0.0000000 0.3152327 0.3357606 0.3574020
 F(21)=   0.0000000 0.0000000 0.0000000 0.3357607 0.3574022
 F(22)=   0.0000000 0.0000000 0.0000000 0.3357607 0.3574023
 F(23)=   0.0000000 0.0000000 0.0000000 0.0000000 0.3574024
 F(24)=   0.0000000 0.0000000 0.0000000 0.0000000 0.3574025
```

```
TABLE I    PAGE  6
SEQUENCE OF ARRIVAL RATES (FINITE LIMITING POINT)
        (ZEROS : NO ENTRY)

 F( 0)=   0.2599998 0.2699998 0.2799998 0.2899998 0.2999998
 F( 1)=   0.3195263 0.3338872 0.3483802 0.3630032 0.3777542
 F( 2)=   0.3473924 0.3647788 0.3824826 0.4005037 0.4188416
 F( 3)=   0.3619497 0.3814943 0.4015658 0.4221730 0.4433239
 F( 4)=   0.3699176 0.3909944 0.4128091 0.4353874 0.4587543
 F( 5)=   0.3743816 0.3965319 0.4196165 0.4436855 0.4687890
 F( 6)=   0.3769138 0.3998049 0.4238029 0.4489868 0.4754395
 F( 7)=   0.3783600 0.4017552 0.4264013 0.4524097 0.4799005
 F( 8)=   0.3791893 0.4029227 0.4280231 0.4546346 0.4829162
 F( 9)=   0.3796659 0.4036235 0.4290390 0.4560869 0.4849656
 F(10)=   0.3799400 0.4040450 0.4296767 0.4570376 0.4863632
 F(11)=   0.3800979 0.4042987 0.4300774 0.4576610 0.4873184
 F(12)=   0.3801889 0.4044514 0.4303296 0.4580703 0.4879724
 F(13)=   0.3802413 0.4045435 0.4304883 0.4583392 0.4884206
 F(14)=   0.3802715 0.4045990 0.4305882 0.4585160 0.4887280
 F(15)=   0.3802889 0.4046324 0.4306512 0.4586322 0.4889390
 F(16)=   0.3802989 0.4046525 0.4306908 0.4587087 0.4890838
 F(17)=   0.3803047 0.4046647 0.4307158 0.4587590 0.4891833
 F(18)=   0.3803080 0.4046720 0.4307315 0.4587921 0.4892516
 F(19)=   0.3803099 0.4046764 0.4307414 0.4588138 0.4892985
 F(20)=   0.3803110 0.4046791 0.4307477 0.4588282 0.4893308
 F(21)=   0.3803117 0.4046807 0.4307516 0.4588376 0.4893529
 F(22)=   0.3803120 0.4046816 0.4307541 0.4588438 0.4893681
 F(23)=   0.3803122 0.4046822 0.4307556 0.4588478 0.4893785
 F(24)=   0.3803123 0.4046826 0.4307566 0.4588505 0.4893857
 F(25)=   0.3803124 0.4046828 0.4307572 0.4588523 0.4893906
 F(26)=   0.3803125 0.4046829 0.4307576 0.4588535 0.4893940
 F(27)=   0.0000000 0.4046829 0.4307579 0.4588543 0.4893963
 F(28)=   0.0000000 0.0000000 0.4307580 0.4588548 0.4893979
 F(29)=   0.0000000 0.0000000 0.4307581 0.4588551 0.4893990
 F(30)=   0.0000000 0.0000000 0.4307582 0.4588553 0.4893998
 F(31)=   0.0000000 0.0000000 0.0000000 0.4588555 0.4894003
 F(32)=   0.0000000 0.0000000 0.0000000 0.4588555 0.4894007
 F(33)=   0.0000000 0.0000000 0.0000000 0.0000000 0.4894009
 F(34)=   0.0000000 0.0000000 0.0000000 0.0000000 0.4894011
 F(35)=   0.0000000 0.0000000 0.0000000 0.0000000 0.4894012
 F(36)=   0.0000000 0.0000000 0.0000000 0.0000000 0.4894013
```

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TABLE I     PAGE  7
SEQUENCE OF ARRIVAL RATES (FINITE LIMITING POINT)
        (ZEROS : NO ENTRY)


 F( 0)=   0.3099998 0.3199998 0.3299997 0.3399997 0.3499997
 F( 1)=   0.3926311 0.4076319 0.4227548 0.4379976 0.4533587
 F( 2)=   0.4374958 0.4564652 0.4757490 0.4953455 0.5152534
 F( 3)=   0.4650258 0.4872856 0.5101090 0.5335013 0.5574669
 F( 4)=   0.4829341 0.5079499 0.5338239 0.5605771 0.5882289
 F( 5)=   0.4949777 0.5223020 0.5508115 0.5805548 0.6115792
 F( 6)=   0.5032467 0.5324965 0.5632784 0.5956832 0.6298012
 F( 7)=   0.5090013 0.5398476 0.5725815 0.6073508 0.6443073
 F( 8)=   0.5130426 0.5452042 0.5796067 0.6164705 0.6560289
 F( 9)=   0.5158985 0.5491363 0.5849579 0.6236708 0.6656100
 F(10)=   0.5179254 0.5520382 0.5890604 0.6293996 0.6735129
 F(11)=   0.5193684 0.5541881 0.5922210 0.6339849 0.6800788
 F(12)=   0.5203978 0.5557855 0.5946648 0.6376723 0.6855662
 F(13)=   0.5211334 0.5569747 0.5965597 0.6406487 0.6901743
 F(14)=   0.5216596 0.5578615 0.5980322 0.6430582 0.6940594
 F(15)=   0.5220363 0.5585235 0.5991784 0.6450136 0.6973459
 F(16)=   0.5223061 0.5590182 0.6000718 0.6466035 0.7001336
 F(17)=   0.5224994 0.5593880 0.6007688 0.6478981 0.7025037
 F(18)=   0.5226380 0.5596647 0.6013130 0.6489537 0.7045227
 F(19)=   0.5227373 0.5598717 0.6017382 0.6498152 0.7062455
 F(20)=   0.5228085 0.5600266 0.6020706 0.6505190 0.7077176
 F(21)=   0.5228596 0.5601426 0.6023304 0.6510942 0.7089770
 F(22)=   0.5228962 0.5602294 0.6025336 0.6515647 0.7100555
 F(23)=   0.5229225 0.5602944 0.6026927 0.6519496 0.7109799
 F(24)=   0.5229413 0.5603431 0.6028171 0.6522647 0.7117729
 F(25)=   0.5229548 0.5603796 0.6029145 0.6525226 0.7124534
 F(26)=   0.5229645 0.5604069 0.6029907 0.6527339 0.7130379
 F(27)=   0.5229715 0.5604274 0.6030504 0.6529070 0.7135401
 F(28)=   0.5229764 0.5604427 0.6030971 0.6530488 0.7139717
 F(29)=   0.5229800 0.5604542 0.6031336 0.6531650 0.7143428
 F(30)=   0.5229825 0.5604628 0.6031622 0.6532602 0.7146619
 F(31)=   0.5229844 0.5604693 0.6031846 0.6533382 0.7149364
 F(32)=   0.5229857 0.5604741 0.6032022 0.6534021 0.7151726
 F(33)=   0.5229867 0.5604777 0.6032159 0.6534546 0.7153760
 F(34)=   0.5229873 0.5604804 0.6032267 0.6534976 0.7155510
 F(35)=   0.5229878 0.5604824 0.6032351 0.6535328 0.7157016
 F(36)=   0.5229881 0.5604839 0.6032417 0.6535617 0.7158313
 F(37)=   0.5229884 0.5604850 0.6032469 0.6535853 0.7159429
 F(38)=   0.5229886 0.5604858 0.6032509 0.6536047 0.7160391
 F(39)=   0.5229887 0.5604865 0.6032540 0.6536206 0.7161220
 F(40)=   0.5229888 0.5604870 0.6032565 0.6536337 0.7161933
 F(41)=   0.5229889 0.5604873 0.6032584 0.6536443 0.7162548
 F(42)=   0.0000000 0.5604876 0.6032599 0.6536531 0.7163077
 F(43)=   0.0000000 0.5604879 0.6032611 0.6536603 0.7163532
 F(44)=   0.0000000 0.5604880 0.6032621 0.6536662 0.7163925
 F(45)=   0.0000000 0.5604882 0.6032628 0.6536710 0.7164264
 F(46)=   0.0000000 0.5604882 0.6032633 0.6536750 0.7164555
 F(47)=   0.0000000 0.0000000 0.6032637 0.6536782 0.7164806
 F(48)=   0.0000000 0.0000000 0.6032641 0.6536809 0.7165022
 F(49)=   0.0000000 0.0000000 0.6032644 0.6536830 0.7165209
 F(50)=   0.0000000 0.0000000 0.6032646 0.6536848 0.7165369
 F(51)=   0.0000000 0.0000000 0.6032648 0.6536863 0.7165508
 F(52)=   0.0000000 0.0000000 0.6032649 0.6536875 0.7165626
 F(53)=   0.0000000 0.0000000 0.6032650 0.6536885 0.7165729
 F(54)=   0.0000000 0.0000000 0.6032651 0.6536893 0.7165818
 F(55)=   0.0000000 0.0000000 0.0000000 0.6536900 0.7165893
 F(56)=   0.0000000 0.0000000 0.0000000 0.6536905 0.7165959
 F(57)=   0.0000000 0.0000000 0.0000000 0.6536909 0.7166016
 F(58)=   0.0000000 0.0000000 0.0000000 0.6536913 0.7166065
 F(59)=   0.0000000 0.0000000 0.0000000 0.6536916 0.7166107
 F(60)=   0.0000000 0.0000000 0.0000000 0.6536918 0.7166142
 F(61)=   0.0000000 0.0000000 0.0000000 0.6536921 0.7166174
```

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F(62)=   0.0000000 0.0000000 0.0000000 0.6536922 0.7166201
F(63)=   0.0000000 0.0000000 0.0000000 0.6536924 0.7166224
F(64)=   0.0000000 0.0000000 0.0000000 0.6536925 0.7166244
F(65)=   0.0000000 0.0000000 0.0000000 0.6536926 0.7166262
F(66)=   0.0000000 0.0000000 0.0000000 0.6536927 0.7166276
F(67)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166288
F(68)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166299
F(69)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166309
F(70)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166317
F(71)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166324
F(72)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166330
F(73)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166336
F(74)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166340
F(75)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166344
F(76)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166348
F(77)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166350
F(78)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166353
F(79)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166355
F(80)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166356
F(81)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166358
F(82)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166359
F(83)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166361
F(84)=   0.0000000 0.0000000 0.0000000 0.0000000 0.7166361
```

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TABLE I    PAGE  8
SEQUENCE OF ARRIVAL RATES (FINITE LIMITING POINT)
        (ZEROS : NO ENTRY)


 F(  0)=   0.3599997 0.0000000 0.0000000 0.0000000 0.0000000
 F(  1)=   0.4688361 0.0000000 0.0000000 0.0000000 0.0000000
 F(  2)=   0.5354709 0.0000000 0.0000000 0.0000000 0.0000000
 F(  3)=   0.5820094 0.0000000 0.0000000 0.0000000 0.0000000
 F(  4)=   0.6167971 0.0000000 0.0000000 0.0000000 0.0000000
 F(  5)=   0.6439298 0.0000000 0.0000000 0.0000000 0.0000000
 F(  6)=   0.6657219 0.0000000 0.0000000 0.0000000 0.0000000
 F(  7)=   0.6836056 0.0000000 0.0000000 0.0000000 0.0000000
 F(  8)=   0.6985255 0.0000000 0.0000000 0.0000000 0.0000000
 F(  9)=   0.7111359 0.0000000 0.0000000 0.0000000 0.0000000
 F( 10)=   0.7219067 0.0000000 0.0000000 0.0000000 0.0000000
 F( 11)=   0.7311860 0.0000000 0.0000000 0.0000000 0.0000000
 F( 12)=   0.7392382 0.0000000 0.0000000 0.0000000 0.0000000
 F( 13)=   0.7462683 0.0000000 0.0000000 0.0000000 0.0000000
 F( 14)=   0.7524379 0.0000000 0.0000000 0.0000000 0.0000000
 F( 15)=   0.7578766 0.0000000 0.0000000 0.0000000 0.0000000
 F( 16)=   0.7626894 0.0000000 0.0000000 0.0000000 0.0000000
 F( 17)=   0.7669629 0.0000000 0.0000000 0.0000000 0.0000000
 F( 18)=   0.7707686 0.0000000 0.0000000 0.0000000 0.0000000
 F( 19)=   0.7741665 0.0000000 0.0000000 0.0000000 0.0000000
 F( 20)=   0.7772074 0.0000000 0.0000000 0.0000000 0.0000000
 F( 21)=   0.7799343 0.0000000 0.0000000 0.0000000 0.0000000
 F( 22)=   0.7823839 0.0000000 0.0000000 0.0000000 0.0000000
 F( 23)=   0.7845881 0.0000000 0.0000000 0.0000000 0.0000000
 F( 24)=   0.7865743 0.0000000 0.0000000 0.0000000 0.0000000
 F( 25)=   0.7883663 0.0000000 0.0000000 0.0000000 0.0000000
 F( 26)=   0.7899851 0.0000000 0.0000000 0.0000000 0.0000000
 F( 27)=   0.7914488 0.0000000 0.0000000 0.0000000 0.0000000
 F( 28)=   0.7927737 0.0000000 0.0000000 0.0000000 0.0000000
 F( 29)=   0.7939737 0.0000000 0.0000000 0.0000000 0.0000000
 F( 30)=   0.7950616 0.0000000 0.0000000 0.0000000 0.0000000
 F( 31)=   0.7960486 0.0000000 0.0000000 0.0000000 0.0000000
 F( 32)=   0.7969444 0.0000000 0.0000000 0.0000000 0.0000000
 F( 33)=   0.7977581 0.0000000 0.0000000 0.0000000 0.0000000
 F( 34)=   0.7984975 0.0000000 0.0000000 0.0000000 0.0000000
 F( 35)=   0.7991697 0.0000000 0.0000000 0.0000000 0.0000000
 F( 36)=   0.7997810 0.0000000 0.0000000 0.0000000 0.0000000
 F( 37)=   0.8003372 0.0000000 0.0000000 0.0000000 0.0000000
 F( 38)=   0.8008435 0.0000000 0.0000000 0.0000000 0.0000000
 F( 39)=   0.8013044 0.0000000 0.0000000 0.0000000 0.0000000
 F( 40)=   0.8017242 0.0000000 0.0000000 0.0000000 0.0000000
 F( 41)=   0.8021066 0.0000000 0.0000000 0.0000000 0.0000000
 F( 42)=   0.8024550 0.0000000 0.0000000 0.0000000 0.0000000
 F( 43)=   0.8027726 0.0000000 0.0000000 0.0000000 0.0000000
 F( 44)=   0.8030620 0.0000000 0.0000000 0.0000000 0.0000000
 F( 45)=   0.8033259 0.0000000 0.0000000 0.0000000 0.0000000
 F( 46)=   0.8035665 0.0000000 0.0000000 0.0000000 0.0000000
 F( 47)=   0.8037860 0.0000000 0.0000000 0.0000000 0.0000000
 F( 48)=   0.8039861 0.0000000 0.0000000 0.0000000 0.0000000
 F( 49)=   0.8041687 0.0000000 0.0000000 0.0000000 0.0000000
 F( 50)=   0.8043353 0.0000000 0.0000000 0.0000000 0.0000000
 F( 51)=   0.8044873 0.0000000 0.0000000 0.0000000 0.0000000
 F( 52)=   0.8046260 0.0000000 0.0000000 0.0000000 0.0000000
 F( 53)=   0.8047526 0.0000000 0.0000000 0.0000000 0.0000000
 F( 54)=   0.8048681 0.0000000 0.0000000 0.0000000 0.0000000
 F( 55)=   0.8049735 0.0000000 0.0000000 0.0000000 0.0000000
 F( 56)=   0.8050697 0.0000000 0.0000000 0.0000000 0.0000000
 F( 57)=   0.8051576 0.0000000 0.0000000 0.0000000 0.0000000
 F( 58)=   0.8052378 0.0000000 0.0000000 0.0000000 0.0000000
 F( 59)=   0.8053111 0.0000000 0.0000000 0.0000000 0.0000000
 F( 60)=   0.8053779 0.0000000 0.0000000 0.0000000 0.0000000
 F( 61)=   0.8054389 0.0000000 0.0000000 0.0000000 0.0000000
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F(62)=   0.8054946 0.0000000 0.0000000 0.0000000 0.0000000
F(63)=   0.8055455 0.0000000 0.0000000 0.0000000 0.0000000
F(64)=   0.8055919 0.0000000 0.0000000 0.0000000 0.0000000
F(65)=   0.8056344 0.0000000 0.0000000 0.0000000 0.0000000
F(66)=   0.8056731 0.0000000 0.0000000 0.0000000 0.0000000
F(67)=   0.8057085 0.0000000 0.0000000 0.0000000 0.0000000
F(68)=   0.8057408 0.0000000 0.0000000 0.0000000 0.0000000
F(69)=   0.8057703 0.0000000 0.0000000 0.0000000 0.0000000
F(70)=   0.8057972 0.0000000 0.0000000 0.0000000 0.0000000
F(71)=   0.8058218 0.0000000 0.0000000 0.0000000 0.0000000
F(72)=   0.8058442 0.0000000 0.0000000 0.0000000 0.0000000
F(73)=   0.8058648 0.0000000 0.0000000 0.0000000 0.0000000
F(74)=   0.8058835 0.0000000 0.0000000 0.0000000 0.0000000
F(75)=   0.8059006 0.0000000 0.0000000 0.0000000 0.0000000
F(76)=   0.8059162 0.0000000 0.0000000 0.0000000 0.0000000
F(77)=   0.8059305 0.0000000 0.0000000 0.0000000 0.0000000
F(78)=   0.8059435 0.0000000 0.0000000 0.0000000 0.0000000
F(79)=   0.8059555 0.0000000 0.0000000 0.0000000 0.0000000
F(80)=   0.8059663 0.0000000 0.0000000 0.0000000 0.0000000
F(81)=   0.8059763 0.0000000 0.0000000 0.0000000 0.0000000
F(82)=   0.8059853 0.0000000 0.0000000 0.0000000 0.0000000
F(83)=   0.8059936 0.0000000 0.0000000 0.0000000 0.0000000
F(84)=   0.8060012 0.0000000 0.0000000 0.0000000 0.0000000
F(85)=   0.8060081 0.0000000 0.0000000 0.0000000 0.0000000
F(86)=   0.8060144 0.0000000 0.0000000 0.0000000 0.0000000
F(87)=   0.8060202 0.0000000 0.0000000 0.0000000 0.0000000
F(88)=   0.8060254 0.0000000 0.0000000 0.0000000 0.0000000
F(89)=   0.8060303 0.0000000 0.0000000 0.0000000 0.0000000
F(90)=   0.8060347 0.0000000 0.0000000 0.0000000 0.0000000
F(91)=   0.8060387 0.0000000 0.0000000 0.0000000 0.0000000
F(92)=   0.8060424 0.0000000 0.0000000 0.0000000 0.0000000
F(93)=   0.8060457 0.0000000 0.0000000 0.0000000 0.0000000
F(94)=   0.8060488 0.0000000 0.0000000 0.0000000 0.0000000
F(95)=   0.8060516 0.0000000 0.0000000 0.0000000 0.0000000
F(96)=   0.8060541 0.0000000 0.0000000 0.0000000 0.0000000
F(97)=   0.8060564 0.0000000 0.0000000 0.0000000 0.0000000
F(98)=   0.8060586 0.0000000 0.0000000 0.0000000 0.0000000
F(99)=   0.8060606 0.0000000 0.0000000 0.0000000 0.0000000
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TABLE II   PAGE  1
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
        (ZEROS : NO ENTRY)


 F( 0)=   0.3700000 0.3800000 0.3900000 0.4000000 0.4100000
 F( 1)=   0.4844283 0.5001326 0.5159478 0.5318719 0.5479034
 F( 2)=   0.5559965 0.5768270 0.5979607 0.6193949 0.6411271
 F( 3)=   0.6071323 0.6328361 0.6591229 0.6859931 0.7134464
 F( 4)=   0.6462990 0.6767473 0.7081554 0.7405339 0.7738914
 F( 5)=   0.6776503 0.7127786 0.7493520 0.7874035 0.8269612
 F( 6)=   0.7035335 0.7433175 0.7851537 0.8291154 0.8752678
 F( 7)=   0.7254015 0.7698449 0.8170847 0.8672602 0.9204996
 F( 8)=   0.7442127 0.7933414 0.8461640 0.9029233 0.9638461
 F( 9)=   0.7606312 0.8144886 0.8731109 0.9368937 1.0062120
 F(10)=   0.7751340 0.8337795 0.8984559 0.9697784 1.0483390
 F(11)=   0.7880747 0.8515821 0.9226065 1.0020690 1.0908760
 F(12)=   0.7997218 0.8681785 0.9458873 1.0341900 1.1344270
 F(13)=   0.8102837 0.8837907 0.9685663 1.0665200 1.1795870
 F(14)=   0.8199247 0.8985963 0.9908724 1.0994190 1.2269750
 F(15)=   0.8287767 0.9127411 1.0130080 1.1332410 1.2772530
 F(16)=   0.8369468 0.9263464 1.0351590 1.1683510 1.3311510
 F(17)=   0.8445232 0.9395152 1.0575010 1.2051340 1.3894970
 F(18)=   0.8515790 0.9523365 1.0802060 1.2440130 1.4532330
 F(19)=   0.8581757 0.9648884 1.1034480 1.2854570 1.5234470
 F(20)=   0.8643652 0.9772411 1.1274070 1.3299970 1.6013980
 F(21)=   0.8701918 0.9894584 1.1522710 1.3782410 1.6885320
 F(22)=   0.8756935 1.0015990 1.1782470 1.4308940 1.7865070
 F(23)=   0.8809032 1.0137190 1.2055610 1.4887740 1.8971880
 F(24)=   0.8858495 1.0258740 1.2344660 1.5528320 2.0226280
 F(25)=   0.8905573 1.0381150 1.2652480 1.6241780 1.7550200
 F(26)=   0.8950486 1.0504960 1.2982330 1.7040950 1.4515700
 F(27)=   0.8993427 1.0630690 1.3337980 1.7940560 1.1116090
 F(28)=   0.9034569 1.0758910 1.3723760 1.8957320 0.7458577
 F(29)=   0.9074063 1.0890180 1.4144720 2.0109770 0.3920770
 F(30)=   0.9112047 1.1025120 1.4606790 2.1417920 0.1271694
 F(31)=   0.9148643 1.1164370 1.5116850 1.8902510 0.0151860
 F(32)=   0.9183962 1.1308660 1.5683000 1.6047590 0.0002289
 F(33)=   0.9218104 1.1458740 1.6314660 1.2823010 0.0000001
 F(34)=   0.9251159 1.1615470 1.7022810 0.9265939 0.0000000
 F(35)=   0.9283209 1.1779810 1.7820110 0.5597555 0.0000000
 F(36)=   0.9314329 1.1952810 1.8720980 0.2399399 0.0000000
 F(37)=   0.9344588 1.2135650 1.9741670 0.0511851 0.0000000
 F(38)=   0.9374049 1.2329710 2.0900010 0.0025540 0.0000000
 F(39)=   0.9402769 1.2536510 1.8314940 0.0000065 0.0000000
 F(40)=   0.9430803 1.2757830 1.5381360 0.0000000 0.0000000
 F(41)=   0.9458199 1.2995690 1.2077730 0.0000000 0.0000000
 F(42)=   0.9485003 1.3252420 0.8468156 0.0000000 0.0000000
 F(43)=   0.9511258 1.3530730 0.4837196 0.0000000 0.0000000
 F(44)=   0.9537003 1.3833770 0.1855132 0.0000000 0.0000000
 F(45)=   0.9562276 1.4165210 0.0314115 0.0000000 0.0000000
 F(46)=   0.9587111 1.4529330 0.0009713 0.0000000 0.0000000
 F(47)=   0.9611540 1.4931150 0.0000009 0.0000000 0.0000000
 F(48)=   0.9635594 1.5376530 0.0000000 0.0000000 0.0000000
 F(49)=   0.9659303 1.5872340 0.0000000 0.0000000 0.0000000
 F(50)=   0.9682693 1.6426590 0.0000000 0.0000000 0.0000000
 F(51)=   0.9705790 1.7048600 0.0000000 0.0000000 0.0000000
 F(52)=   0.9728619 1.7749190 0.0000000 0.0000000 0.0000000
 F(53)=   0.9751204 1.8540740 0.0000000 0.0000000 0.0000000
 F(54)=   0.9773567 1.9437290 0.0000000 0.0000000 0.0000000
 F(55)=   0.9795730 2.0454460 0.0000000 0.0000000 0.0000000
 F(56)=   0.9817714 2.1609230 0.0000000 0.0000000 0.0000000
 F(57)=   0.9839538 1.9119430 0.0000000 0.0000000 0.0000000
 F(58)=   0.9861223 1.6293710 0.0000000 0.0000000 0.0000000
 F(59)=   0.9882786 1.3099280 0.0000000 0.0000000 0.0000000
 F(60)=   0.9904246 0.9564587 0.0000000 0.0000000 0.0000000
 F(61)=   0.9925621 0.5889383 0.0000000 0.0000000 0.0000000
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F(62)=   0.9946929 0.2621269 0.0000000 0.0000000 0.0000000
F(63)=   0.9968187 0.0604430 0.0000000 0.0000000 0.0000000
F(64)=   0.9989411 0.0035451 0.0000000 0.0000000 0.0000000
F(65)=   1.0010610 0.0000125 0.0000000 0.0000000 0.0000000
F(66)=   1.0031810 0.0000000 0.0000000 0.0000000 0.0000000
F(67)=   1.0053030 0.0000000 0.0000000 0.0000000 0.0000000
F(68)=   1.0074280 0.0000000 0.0000000 0.0000000 0.0000000
F(69)=   1.0095580 0.0000000 0.0000000 0.0000000 0.0000000
F(70)=   1.0116950 0.0000000 0.0000000 0.0000000 0.0000000
F(71)=   1.0138400 0.0000000 0.0000000 0.0000000 0.0000000
F(72)=   1.0159950 0.0000000 0.0000000 0.0000000 0.0000000
F(73)=   1.0181620 0.0000000 0.0000000 0.0000000 0.0000000
F(74)=   1.0203420 0.0000000 0.0000000 0.0000000 0.0000000
F(75)=   1.0225380 0.0000000 0.0000000 0.0000000 0.0000000
F(76)=   1.0247500 0.0000000 0.0000000 0.0000000 0.0000000
F(77)=   1.0269810 0.0000000 0.0000000 0.0000000 0.0000000
F(78)=   1.0292320 0.0000000 0.0000000 0.0000000 0.0000000
F(79)=   1.0315070 0.0000000 0.0000000 0.0000000 0.0000000
F(80)=   1.0338060 0.0000000 0.0000000 0.0000000 0.0000000
F(81)=   1.0361320 0.0000000 0.0000000 0.0000000 0.0000000
F(82)=   1.0384870 0.0000000 0.0000000 0.0000000 0.0000000
F(83)=   1.0408730 0.0000000 0.0000000 0.0000000 0.0000000
F(84)=   1.0432920 0.0000000 0.0000000 0.0000000 0.0000000
F(85)=   1.0457470 0.0000000 0.0000000 0.0000000 0.0000000
F(86)=   1.0482410 0.0000000 0.0000000 0.0000000 0.0000000
F(87)=   1.0507760 0.0000000 0.0000000 0.0000000 0.0000000
F(88)=   1.0533550 0.0000000 0.0000000 0.0000000 0.0000000
F(89)=   1.0559800 0.0000000 0.0000000 0.0000000 0.0000000
F(90)=   1.0586550 0.0000000 0.0000000 0.0000000 0.0000000
F(91)=   1.0613840 0.0000000 0.0000000 0.0000000 0.0000000
F(92)=   1.0641690 0.0000000 0.0000000 0.0000000 0.0000000
F(93)=   1.0670150 0.0000000 0.0000000 0.0000000 0.0000000
F(94)=   1.0699260 0.0000000 0.0000000 0.0000000 0.0000000
F(95)=   1.0729040 0.0000000 0.0000000 0.0000000 0.0000000
F(96)=   1.0759560 0.0000000 0.0000000 0.0000000 0.0000000
F(97)=   1.0790850 0.0000000 0.0000000 0.0000000 0.0000000
F(98)=   1.0822970 0.0000000 0.0000000 0.0000000 0.0000000
F(99)=   1.0855970 0.0000000 0.0000000 0.0000000 0.0000000
```

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TABLE II    PAGE  2
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
         (ZEROS : NO ENTRY)


 F( 0)=   0.4200000 0.4299999 0.4399999 0.4499999 0.4599999
 F( 1)=   0.5640402 0.5802810 0.5966238 0.6130672 0.6296094
 F( 2)=   0.6631544 0.6854738 0.7080823 0.7309767 0.7541535
 F( 3)=   0.7414815 0.7700968 0.7992895 0.8290563 0.8593931
 F( 4)=   0.8082341 0.8435664 0.8798902 0.9172055 0.9555097
 F( 5)=   0.8680491 0.9106863 0.9548867 1.0006580 1.0480050
 F( 6)=   0.9236684 0.9743645 1.0273920 1.0827770 1.1405350
 F( 7)=   0.9769168 1.0366070 1.0996470 1.1660900 1.2359640
 F( 8)=   1.0291360 1.0989680 1.1734770 1.2527570 1.3368490
 F( 9)=   1.0814090 1.1627750 1.2505320 1.3448230 1.4456950
 F(10)=   1.1346840 1.2292710 1.3324380 1.4443810 1.5651140
 F(11)=   1.1898560 1.2997020 1.4208960 1.5536630 1.6979030
 F(12)=   1.2478240 1.3753860 1.5177520 1.6751070 1.8470720
 F(13)=   1.3095370 1.4577670 1.6250540 1.8113780 2.0157900
 F(14)=   1.3760340 1.5484620 1.7450790 1.9653460 1.7472570
 F(15)=   1.4484760 1.6492970 1.8803320 1.6899860 1.4427940
 F(16)=   1.5281870 1.7623260 2.0335070 1.3781460 1.1019100
 F(17)=   1.6166790 1.8898300 1.7673700 1.0307900 0.7358165
 F(18)=   1.7156750 2.0342800 1.4655350 0.6630817 0.3832765
 F(19)=   1.8271230 1.7682480 1.1270620 0.3214216 0.1220258
 F(20)=   1.9531820 1.4665280 0.7619131 0.0883532 0.0140177
 F(21)=   2.0961760 1.1281630 0.4062729 0.0074714 0.0001951
 F(22)=   1.8385010 0.7630591 0.1356430 0.0000556 0.0000000
 F(23)=   1.5460770 0.4072920 0.0172057 0.0000000 0.0000000
 F(24)=   1.2166350 0.1362596 0.0002935 0.0000000 0.0000000
 F(25)=   0.8562376 0.0173573 0.0000001 0.0000000 0.0000000
 F(26)=   0.4925446 0.0002987 0.0000000 0.0000000 0.0000000
 F(27)=   0.1915656 0.0000001 0.0000000 0.0000000 0.0000000
 F(28)=   0.0333965 0.0000000 0.0000000 0.0000000 0.0000000
 F(29)=   0.0010969 0.0000000 0.0000000 0.0000000 0.0000000
 F(30)=   0.0000012 0.0000000 0.0000000 0.0000000 0.0000000
 F(31)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 F(32)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

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TABLE II    PAGE   3
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
        (ZEROS : NO ENTRY)


 F( 0)=   0.4699999 0.4799999 0.4899999 0.4999999 0.5099999
 F( 1)=   0.6462488 0.6629838 0.6798128 0.6967344 0.7137470
 F( 2)=   0.7776093 0.8013405 0.8253435 0.8496146 0.8741497
 F( 3)=   0.8902952 0.9217573 0.9537733 0.9863365 1.0194390
 F( 4)=   0.9947982 1.0350630 1.0762980 1.1184910 1.1616280
 F( 5)=   1.0969220 1.1474040 1.1994360 1.2529980 1.3080660
 F( 6)=   1.2006630 1.2631490 1.3279680 1.3950820 1.4644400
 F( 7)=   1.3092690 1.3859770 1.4660360 1.5493620 1.6358490
 F( 8)=   1.4257440 1.5193720 1.6176180 1.7203010 1.8272060
 F( 9)=   1.5530940 1.6668580 1.7867290 1.9123440 1.5332770
 F(10)=   1.6944710 1.8320890 1.4874390 1.6298270 1.2023540
 F(11)=   1.8532010 1.5388100 1.1513500 1.3104410 0.8410639
 F(12)=   1.5627400 1.2085250 0.7872831 0.9570137 0.4783539
 F(13)=   1.2352500 0.8476149 0.4290067 0.5894840 0.1818688
 F(14)=   0.8760867 0.4844665 0.1496566 0.2625483 0.0302428
 F(15)=   0.5112761 0.1860226 0.0208017 0.0606252 0.0009009
 F(16)=   0.2046486 0.0315765 0.0004282 0.0035662 0.0000008
 F(17)=   0.0378736 0.0009815 0.0000002 0.0000127 0.0000000
 F(18)=   0.0014076 0.0000010 0.0000000 0.0000000 0.0000000
 F(19)=   0.0000020 0.0000000 0.0000000 0.0000000 0.0000000
 F(20)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 F(21)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

```
TABLE II    PAGE  4
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
        (ZEROS : NO ENTRY)


 F( 0)=  0.5199999 0.5299999 0.5399998 0.5499998 0.5599998
 F( 1)=  0.7308490 0.7480391 0.7653157 0.7826773 0.8001226
 F( 2)=  0.8989451 0.9239967 0.9493006 0.9748524 1.0006470
 F( 3)=  1.0530740 1.0872340 1.1219100 1.1570910 1.1927670
 F( 4)=  1.2056950 1.2506750 1.2965510 1.3433010 1.3909040
 F( 5)=  1.3646080 1.4225920 1.4819790 1.5427220 1.6047760
 F( 6)=  1.5359760 1.6096210 1.6852900 1.7628890 1.8423200
 F( 7)=  1.7253630 1.8177540 1.3728530 1.4604680 1.5504050
 F( 8)=  1.4180620 1.5225690 1.0249940 1.1214530 1.2214670
 F( 9)=  1.0746310 1.1904200 0.6572285 0.7560772 0.8613828
 F(10)=  0.7077272 0.8284212 0.3165963 0.4010954 0.4973819
 F(11)=  0.3589855 0.4666179 0.0859163 0.1325275 0.1949137
 F(12)=  0.1082756 0.1739926 0.0070734 0.0164495 0.0345181
 F(13)=  0.0111112 0.0277861 0.0000499 0.0002684 0.0011712
 F(14)=  0.0001228 0.0007614 0.0000000 0.0000001 0.0000014
 F(15)=  0.0000000 0.0000006 0.0000000 0.0000000 0.0000000
 F(16)=  0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 F(17)=  0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

```
TABLE II    PAGE  5
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
        (ZEROS : NO ENTRY)


 F( 0)=  0.5699998 0.5799998 0.5899998 0.5999998 0.6099998
 F( 1)=  0.8176501 0.8352586 0.8529465 0.8707126 0.8885556
 F( 2)=  1.0266820 1.0529540 1.0794560 1.1061840 1.1331360
 F( 3)=  1.2289310 1.2655720 1.3026780 1.3402370 1.3782410
 F( 4)=  1.4393390 1.4885810 1.5386050 1.5893850 1.6408940
 F( 5)=  1.6680930 1.7326180 1.7982950 1.2650690 1.3228770
 F( 6)=  1.3534800 1.4262550 1.5005320 0.9080415 0.9705060
 F( 7)=  1.0038220 1.0836580 1.1658950 0.5418162 0.6027898
 F( 8)=  0.6359456 0.7169966 0.8025518 0.2266467 0.2728933
 F( 9)=  0.2992539 0.3669471 0.4428611 0.0459634 0.0651740
 F(10)=  0.0773957 0.1127092 0.1584573 0.0020648 0.0041122
 F(11)=  0.0057642 0.0120136 0.0232204 0.0000043 0.0000169
 F(12)=  0.0000331 0.0001435 0.0005330 0.0000000 0.0000000
 F(13)=  0.0000000 0.0000000 0.0000003 0.0000000 0.0000000
 F(14)=  0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 F(15)=  0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

```
TABLE II   PAGE  6
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
        (ZEROS : NO ENTRY)

 F( 0)=  0.6199998 0.6299998 0.6399997 0.6499997 0.6599997
 F( 1)=  0.9064740 0.9244667 0.9425324 0.9606698 0.9788776
 F( 2)=  1.1603060 1.1876900 1.2152830 1.2430820 1.2710800
 F( 3)=  1.4166770 1.4555330 1.4947970 1.5344600 1.5745050
 F( 4)=  1.6931050 1.7459910 1.1595230 1.2036730 1.2484090
 F( 5)=  1.3816630 1.4413650 0.7958552 0.8424632 0.8901646
 F( 6)=  1.0346440 1.1003310 0.4367691 0.4796578 0.5246735
 F( 7)=  0.6669804 0.7341841 0.1545636 0.1827519 0.2141987
 F( 8)=  0.3246487 0.3818515 0.0221352 0.0305242 0.0413001
 F( 9)=  0.0899987 0.1212010 0.0004846 0.0009176 0.0016710
 F(10)=  0.0077460 0.0138344 0.0000002 0.0000008 0.0000028
 F(11)=  0.0000598 0.0001901 0.0000000 0.0000000 0.0000000
 F(12)=  0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 F(13)=  0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

```
TABLE II    PAGE  7
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
        (ZEROS : NO ENTRY)


 F( 0)=   0.6699997 0.6799997 0.6899997 0.6999997 0.7099997
 F( 1)=   0.9971547 1.0154990 1.0339110 1.0523890 1.0709310
 F( 2)=   1.2992760 1.3276620 1.3562380 1.3849960 1.4139330
 F( 3)=   1.6149250 1.6557030 1.6968320 1.0382960 1.0700860
 F( 4)=   1.2937070 1.3395330 1.3858650 0.6706803 0.7030694
 F( 5)=   0.9389058 0.9886191 1.0392490 0.3277208 0.3550055
 F( 6)=   0.5717415 0.6207637 0.6716460 0.0915769 0.1060864
 F( 7)=   0.2489697 0.2870823 0.3285242 0.0080138 0.0106779
 F( 8)=   0.0548720 0.0716414 0.0919915 0.0000640 0.0001134
 F( 9)=   0.0029298 0.0049530 0.0080849 0.0000000 0.0000000
 F(10)=   0.0000086 0.0000245 0.0000651 0.0000000 0.0000000
 F(11)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 F(12)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

```
TABLE II    PAGE  8
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
        (ZEROS : NO ENTRY)

 F( 0)=  0.7199997 0.7299997 0.7399997 0.7499996 0.7599996
 F( 1)=  1.0895370 1.1082050 1.1269350 1.1457240 1.1645720
 F( 2)=  1.4430460 1.4723300 1.5017790 1.5313900 1.5611600
 F( 3)=  1.1021880 1.1345910 1.1672820 1.2002490 1.2334840
 F( 4)=  0.7361035 0.7697593 0.8040103 0.8388317 0.8742000
 F( 5)=  0.3835271 0.4132649 0.4441911 0.4762759 0.5094876
 F( 6)=  0.1221712 0.1398955 0.1593123 0.1804647 0.2033857
 F( 7)=  0.0140501 0.0182635 0.0234619 0.0297979 0.0374304
 F( 8)=  0.0001960 0.0003305 0.0005441 0.0008748 0.0013751
 F( 9)=  0.0000000 0.0000001 0.0000003 0.0000008 0.0000019
 F(10)=  0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 F(11)=  0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

```
TABLE II    PAGE  9
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
        (ZEROS : NO ENTRY)

 F(  0)=   0.7699996 0.7799996 0.7899996 0.7999996 0.8099996
 F(  1)=   1.1834780 1.2024420 1.2214610 1.2405350 1.2596630
 F(  2)=   1.5910820 1.6211550 1.6513740 0.8817365 0.9022347
 F(  3)=   1.2669700 1.3007010 1.3346630 0.5166419 0.5362322
 F(  4)=   0.9100857 0.9464672 0.9833176 0.2084544 0.2225642
 F(  5)=   0.5437856 0.5791340 0.6154899 0.0392232 0.0444097
 F(  6)=   0.2280928 0.2545969 0.2828938 0.0015087 0.0019291
 F(  7)=   0.0465195 0.0572260 0.0697051 0.0000023 0.0000037
 F(  8)=   0.0021145 0.0031829 0.0046933 0.0000000 0.0000000
 F(  9)=   0.0000045 0.0000101 0.0000220 0.0000000 0.0000000
 F( 10)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 F( 11)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

```
TABLE II   PAGE 10
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
         (ZEROS : NO ENTRY)

 F(  0)=   0.8199996 0.8299996 0.8399996 0.8499995 0.8599995
 F(  1)=   1.2788440 1.2980770 1.3173610 1.3366950 1.3560790
 F(  2)=   0.9228673 0.9436294 0.9645184 0.9855292 1.0066580
 F(  3)=   0.5561401 0.5763569 0.5968761 0.6176887 0.6387873
 F(  4)=   0.2372393 0.2524778 0.2682787 0.2846376 0.3015509
 F(  5)=   0.0501044 0.0563345 0.0631274 0.0705085 0.0785027
 F(  6)=   0.0024486 0.0030858 0.0038619 0.0048002 0.0059270
 F(  7)=   0.0000060 0.0000095 0.0000149 0.0000230 0.0000350
 F(  8)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 F(  9)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

```
TABLE II    PAGE 11
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
         (ZEROS : NO ENTRY)


 F(  0)=   0.8699995 0.8799995 0.8899995 0.8999995 0.9099995
 F(  1)=   1.3755110 1.3949890 1.4145140 1.4340850 1.4537010
 F(  2)=   1.0279040 1.0492610 1.0707250 1.0922950 1.1139650
 F(  3)=   0.6601657 0.6818137 0.7037238 0.7258899 0.7483006
 F(  4)=   0.3190147 0.3370212 0.3555638 0.3746364 0.3942273
 F(  5)=   0.0871341 0.0964239 0.1063924 0.1170591 0.1284389
 F(  6)=   0.0072710 0.0088634 0.0107380 0.0129312 0.0154811
 F(  7)=   0.0000527 0.0000782 0.0001147 0.0001661 0.0002378
 F(  8)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000001
 F(  9)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

```
TABLE II    PAGE 12
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
         (ZEROS : NO ENTRY)


 F(  0)=   0.9199995 0.9299995 0.9399995 0.9499995 0.9599994
 F(  1)=   1.4733610 1.4930630 1.5128070 1.5325940 1.5524210
 F(  2)=   1.1357330 1.1575970 1.1795500 1.2015920 1.2237190
 F(  3)=   0.7709514 0.7938338 0.8169364 0.8402557 0.8637820
 F(  4)=   0.4143303 0.4349350 0.4560277 0.4776012 0.4996420
 F(  5)=   0.1405480 0.1533986 0.1669990 0.1813597 0.1964853
 F(  6)=   0.0184284 0.0218152 0.0256844 0.0300812 0.0350503
 F(  7)=   0.0003365 0.0004707 0.0006513 0.0008914 0.0012072
 F(  8)=   0.0000001 0.0000002 0.0000004 0.0000008 0.0000015
 F(  9)=   0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 F(10)=    0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

```
TABLE II    PAGE 13
SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)
        (ZEROS : NO ENTRY)

 F( 0)=  0.9699994 0.9799994 0.9899994 0.9999994 1.0099990
 F( 1)=  1.5722870 1.5921930 1.6121370 1.6321190 1.6521360
 F( 2)=  1.2459270 1.2682150 1.2905780 1.3130140 1.3355200
 F( 3)=  0.8875070 0.9114240 0.9355258 0.9598038 0.9842505
 F( 4)=  0.5221374 0.5450759 0.5684447 0.5922296 0.6164171
 F( 5)=  0.2123787 0.2290421 0.2464744 0.2646717 0.2836287
 F( 6)=  0.0406369 0.0468861 0.0538420 0.0615473 0.0700431
 F( 7)=  0.0016183 0.0021476 0.0028223 0.0036739 0.0047382
 F( 8)=  0.0000026 0.0000046 0.0000080 0.0000135 0.0000224
 F( 9)=  0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 F(10)=  0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
```

TABLE III

LIMITING POINTS AND EXPECTED NUMBER OF TRANSMISSIONS

ARRIVAL RATE	NUMBER OF APPROXIMATIONS	LIMITING POINT	EXP NR OF TRANSMISSIONS
0.01	2	0.0101015	1.0101510
0.02	3	0.0204124	1.0206210
0.03	3	0.0309428	1.0314260
0.04	3	0.0417034	1.0425840
0.05	3	0.0527060	1.0541190
0.06	3	0.0639632	1.0660520
0.07	3	0.0754887	1.0784100
0.08	3	0.0872977	1.0912200
0.09	3	0.0994064	1.1045140
0.10	3	0.1118325	1.1183250
0.11	3	0.1245959	1.1326900
0.12	3	0.1377181	1.1476510
0.13	4	0.1512233	1.1632550
0.14	4	0.1651377	1.1795540
0.15	4	0.1794912	1.1966070
0.16	4	0.1943169	1.2144800
0.17	4	0.2096523	1.2332480
0.18	4	0.2255397	1.2529980
0.19	4	0.2420274	1.2738280
0.20	5	0.2591709	1.2958540
0.21	5	0.2770343	1.3192110
0.22	4	0.2956923	1.3440560
0.23	5	0.3152329	1.3705770
0.24	4	0.3357609	1.3990030
0.25	4	0.3574026	1.4296110
0.26	5	0.3803127	1.4627410
0.27	5	0.4046832	1.4988260
0.28	5	0.4307584	1.5384230
0.29	6	0.4588559	1.5822620
0.30	5	0.4894017	1.6313390
0.31	5	0.5229893	1.6870620
0.32	5	0.5604886	1.7515270
0.33	5	0.6032658	1.8280780
0.34	7	0.6536931	1.9226270
0.35	6	0.7166372	2.0475350
0.36	6	0.8060818	2.2391160

1

2
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0
1
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3
4

```
$JOB  (1030.0770WT.ROXO).'LUNEBURG',PAGES=20
C
C     ******************************************************
C
C     SEQUENCE OF ARRIVAL RATES  ( FINITE LIMITING POINT)
C
C     ******************************************************
C
      DIMENSION FM(5,100)
C
    1 FORMAT(' '15X,'F(',I2,')= ',5F10.7)
    2 FORMAT('1'//////,15X,'TABLE I    PAGE ',I2,//,15X,
     A'SEQUENCE OF ARRIVAL RATES (FINITE LIMITING POINT)',
     B//,22X,   ' (ZEROS : NO ENTRY)',///)
    3 FORMAT('1')
C
      FF=0.0
      DO 400 I=1,8
      WRITE(6,2) I
      KM=1
      DO 100 II=1,5
      DO 100 JJ=1,100
  100 FM(II,JJ)=0.0
      DO 200 J=1,5
      FF= FF + 0.01
      IF(FF.GT.0.36) GO TO 200
      F=FF
      K=1
      FM(J,1)=F
   10 FL=F
      K=K+1
      F=FF + F*(1.0 - EXP(-F))
      FM(J,K)=F
      IF(K.GE.100) GO TO 20
      IF((F-FL).LT.0.0000001) GO TO 20
      GO TO  10
   20 CONTINUE
      IF(KM.LT.K) KM=K
  200 CONTINUE
      DO 300 J=1,KM
      JJ=J-1
  300 WRITE(6,1) JJ,(FM(K,J),K=1,5)
  400 CONTINUE
      WRITE(6,3)
 9999 STOP
      END

$GO
```

```
$JOB  (1030.0770WT.ROXO).'LUNEBURG',PAGES=20
C
C     *****************************************************
C
C     SEQUENCE OF ARRIVAL RATES ( INFINITE LIMIT. POINT )
C
C     *****************************************************
C
      DIMENSICN FM(5,100)
C
    1 FORMAT(' '15X,'F(',I2,')= ',5F10.7)
    2 FORMAT('1'//////,15X,'TABLE II   PAGE ',I2,//,15X,
     A'SEQUENCE OF ARRIVAL RATES (INFIN. LIMITING POINT)',
     B//,22X,  ' (ZEROS : NO ENTRY)',///)
    3 FORMAT('1')
C
      FF=0.36
      DO  400 JJ=1,13
      WRITE(6,2) JJ
      KM=1
      DO 100 IJ=1,5
      DO 100 JI=1,100
  100 FM(IJ,JI)=0.0
      DO 200 J=1,5
      K=2
      FF=FF + 0.01
      F=FF
      FM(J,1)= F
      FM(J,2)= FF + F*(1.0 - EXP(-F))
      F=FM(J,2)
      Y=F - FF
   10 Z=FF + F*(1.0 - EXP(-F))
      IF((Z-F).GE.Y) GO TO 30
      K=K+1
      FM(J,K)=Z
      F=Z
      IF(K.GE.100) GO TO 1000
      GO TO 10
   20 F=Z
   30 Z= F*(1.0 - EXP(-F))
      K=K+1
      FM(J,K)=Z
      IF((F-Z).LE.0.0000001.OR.K.GE.100) GO TO 1000
      GO TO 20
 1000 CONTINUE
      IF(KM.LT.K) KM=K
  200 CONTINUE
      DO 300 K=1,KM
      KK=K-1
  300 WRITE(6,1) KK,(FM(J,K),J=1,5)
  400 CONTINUE
      WRITE(6,3)
 9999 STOP
      END

$GO
```

```
$JOB  (1030.0770WT.ROXO).'LUNEBURG',PAGES=20
C
C     *****************************************************
C
C     LIMITING POINTS CALCULATED BY NEWTONS METHOD OF
C     SUCCESSIVE APPROXIMATION
C     EXPECTED NUMBER OF TRANSMISSIONS
C
C
C     *****************************************************
C
    1 FORMAT(' '13X,F6.2,I13,4X,2F11.7)
    2 FORMAT('1'//////,15X,'TABLE III',//,15X,'LIMITING ',
     A'POINTS AND EXPECTED NUMBER OF TRANSMISSIONS',////,15X,
     B'ARRIVAL  NUMBER OF       LIMITING  EXP NR OF',' '/,15X,
     C'RATE     APPROXIMATIONS  POINT     TRANSMISSIONS',
     D///)
    3 FORMAT('1')
C
      WRITE(6,2)
      C=0.0
      U=0.0000001
      DO 100 I=1,36
      K=1
      C=C + 0.01
      F=C
   10 E=EXP(-F)
      G=F*E
      D=(G-C)/(E-G)
      IF(D.LT.U.AND.D.GT.(-U)) GO TO 30
      F=F-D
      K=K+1
      IF(K.GE.300) GO TO 20
      GO TO 10
   20 F=0.0
      FF=0.0
      GO TO 40
   30 FF=EXP(F)
   40 WRITE(6,1) C,K,F,FF
  100 CONTINUE
      WRITE(6,3)
 9999 STOP
      END

$GO
```

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13. ABSTRACT

A queueing model with feedback is proposed as a mathematical representation of a communication system with interference-caused retransmissions. A satellite receives messages of constant length from ground stations. New messages arrive in a Poisson pattern with constant rate. A finite number of channels is available for transmission and channel selection is random among all channels. All messages with overlapping transmission times on the same channel have to be retransmitted. The arrival rate of messages is a composition of the constant arrival rate of the basic Poisson process and the arrival rate of the feedback from retransmissions. The arrival rate is approximated in discrete time steps. A necessary and sufficient condition for a finite limiting point of the resulting sequence of arrival rates is derived. Numberical results for finite limiting points, rate of convergence, and expected number of transmissions have been computed. Examples for sequences of arrival rates have been tabulated.

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